Abstract

Predicting the orbit of a satellite is a fundamental requirement in many areas of aerospace, such as mission planning, satellite geodesy, re-entry prediction, maneuver planning, collision avoidance, formation flying, etc. For near-Earth orbits, the forces due to the non-spherical nature of the Earth and atmospheric drag plays an important role and are mainly responsible to bring the satellite back to the Earth. Thus inclusion of the effect of these perturbing forces becomes important for precise orbit computation of near-Earth orbits. Many transformations have emerged in the literature to stabilize the equations of motion either to reduce the accumulation of local numerical errors or allowing the use of large integration step sizes, or both in the transformed space. One such transformation is known as KS transformation by Kustaanheimo and Stiefel [1], who regularized the nonlinear Kepler equation of motion and reduced it into linear differential equations of a harmonic oscillator of constant frequency. In this paper a detailed analysis is carried out for orbit prediction using KS differential equations by including the non-spherical gravitational potential of the Earth as the perturbing force. Earth’s zonal and tesseral harmonics, which produce bands of constant deviation from the spherical field along lines of latitude and longitude, are included to precisely model the Earth’s gravitational potential. Higher order Earth’s flattening terms are included by utilizing the recurrence relations of associated Legendre polynomial and its derivatives. To know the effectiveness of the theory, the results are compared with the real IRS-1A satellite data and the results of other existing solutions for a long duration of nearly a month time. The comparison shows that the KS method provides one of the most accurate techniques for orbit prediction available at present.

Keywords: Orbit prediction; Earth’s flattening; Semi-major axis; Eccentricity; KS elements

Introduction

A satellite orbiting in space experiences the combined effect of various forces, known as the perturbing forces like the shape of the Earth, the Sun’s radiation, the resistance of the atmosphere, attraction due to the Sun and the Moon, the Earth’s magnetic field, etc. To predict the satellite motion precisely a mathematical model representing the combined effect of these force models must be developed properly for integrating the resulting differential equations of motion. For near Earth’s satellite orbit, the Earth’s flattening perturbations are the dominating forces to affect the motion of the satellite. The high precision in orbit computations becomes necessary due to the availability of the present high accurate satellite tracking system. The classical Newtonian equations of motion, which are non linear are not suitable for long-term integration for computing accurate orbit. Many transformations have emerged in the literature to stabilize the equations of motion either to reduce the accumulation of local numerical errors or allowing the use of large integration step sizes, or both in the transformed space. One such transformation is known as KS transformation by Kustaanheimo and Stiefel [1], who regularized the nonlinear Kepler equation of motion and reduced it into linear differential equations of a harmonic oscillator of constant frequency. The method of KS elements [2] has been found to be a very powerful method for obtaining numerical solution with respect to any type of perturbing forces, as
the equations are less sensitive to roundoff and truncation errors [3]. The equations are everywhere regular comparing to the classical Newtonian equations, which are singular at the collision of two bodies. Numerical studies with these equations were carried out using Earth’s zonal harmonic terms up to J_{12} [4]. Analytical theories with Earth’s oblateness terms up to J_{6} using KS elements equations were also developed [5].

In this paper a detailed study is carried out for orbit prediction using KS differential equations by including the non spherical gravitational potential of the Earth as the perturbing force. Earth’s zonal and tesseral harmonics, which produce bands of constant deviation from the spherical field along lines of latitude and longitude are included to precisely model the Earth’s gravitational potential. Higher order Earth’s flattening terms are included to precisely model the Earth’s gravitational potential. As both bodies are approaching nearer, r reduces smaller and if r = 0, there is singularity in the equations of motion (1). If there is no singularity in the perturbing terms, the value of the potential V and the negative of total energy h will be finite at this vanishing radius. Consequently, from the energy relation (2), the velocity \( \mathbf{\ddot{x}} \) must tend to infinity as r approaches zero. Thus, we are confronted with the fact that the solution of the Newtonian set of equations is singular at r = 0. These equations are therefore, not suitable for numerical integration for small values of distance r.

**First Step of Regularization**

In order to obtain regular functions describing the motion and to avoid other singularities, we must multiply the velocity vector \( \mathbf{\ddot{x}} \) by an appropriate scaling factor which vanishes at collision. Choosing the distance r as the scaling factor, we have \( r \frac{d}{dt} \mathbf{\ddot{x}} = \mathbf{\ddot{x}}', \) where the new variable ‘s’ is called fictitious time. The transformation from ordinary time t to fictitious time s is performed by

\[
\frac{d}{dt} = \frac{1}{r} \frac{d}{ds}
\]

\[(3)\]

In terms of the new independent variables, the basic set of equations (1) and (2) can be written as

\[
\frac{d^2 \mathbf{x}}{ds^2} - \frac{r'}{r} \frac{d \mathbf{x}}{ds} + \frac{K^2}{r} \mathbf{x} = r^2 \left( p \frac{\partial V}{\partial \mathbf{x}} \right), \quad \tau' = r
\]

\[(4)\]

where the prime denotes differentiation with respect to s. Since V is to be regarded as a function of the four arguments \( x_1, x_2, x_3 \) and \( t \), its partial derivative with respect to \( t \) cannot be replaced by differentiation with respect to \( s \). Equations (3) and (4) determines \( x_1, x_2, x_3 \) and \( \tau \) as functions of the independent variable s. So, the physical time \( t \) is to be considered as a fourth coordinate of the particle.
The equation (4) is still singular, since in the denominator \( r \) appears. Nevertheless, the first step of regularization procedures is successful, since the unperturbed solutions are regular functions of the independent variable.

**Second Step of Regularization**

The KS (Kustaanheimo-Stiefel) transformation is given by

\[
X^g = L(\vec{u}^g) \vec{u}^g
\]

where \( L(\vec{u}^g) \) is the extended Le-Civita matrix in 4 dimensions.

\[
L ( \vec{u}^g ) = \begin{bmatrix}
  u_1 & -u_2 & -u_3 & u_4 \\
  u_2 & u_1 & -u_4 & -u_3 \\
  u_3 & u_4 & u_1 & u_2 \\
  u_4 & -u_3 & u_2 & -u_1
\end{bmatrix}
\]

and \( \vec{u}^g = (u_1, u_2, u_3, u_4) \) is the position vector in the four dimensional u-space.

**New Time Element**

Introducing a new time element \( \tau \) for perturbed elliptic motion \( (h > 0) \) defined by

\[
\tau = t + \frac{1}{2} \frac{h}{\omega^2} (\vec{u}^g, \vec{u}'^g), \quad \text{where the energy } h = 2 \omega^2
\]

Utilizing the KS transformation (6) in (4) and (7), we get the new sets of equations.

\[
\vec{u}^g '' + \frac{h}{2} \vec{u}^g = -\frac{V}{2} \vec{u}^g - \frac{r}{4} \left( \frac{\partial V}{\partial u^g} - L^T \vec{P} \right)
\]

\[
\tau' = \frac{1}{2} h (K - 2rV) - \frac{r}{4 h} \left( \vec{u}^g, \frac{\partial V}{\partial u^g} - L^T \vec{P} \right) - \frac{h'}{2} (\vec{u}^g, \vec{u}'^g)
\]

The physical time can be computed from (7) as

\[
t = \tau - \frac{1}{h} (\vec{u}^g, \vec{u}'^g).
\]

**A Set of Regular Elements**

Introducing generalized eccentric anomaly of a pure elliptic motion as

\[
E = 2 \omega s, \quad \left( \omega = \sqrt{\frac{h}{2}} \right)
\]

Then

\[
E' = 2 \omega
\]

Using the above relationships the Equations of motion (8) and (9) are transformed in [7].

\[
0 = \frac{d^2 \vec{u}^g}{dE^2} + 4 \omega \frac{d \omega}{dE} \frac{d \vec{u}^g}{dE} + \frac{4 \omega^2}{2} \vec{u}^g = \frac{V}{2} \vec{u}^g - \frac{r}{4} \left( \frac{\partial V}{\partial u^g} - 2 L^T \vec{P} \right)
\]

\[
\frac{d^2 \vec{u}^g}{dE^2} + \frac{1}{4} \vec{u}^g = -\frac{V}{2} \vec{u}^g + \frac{r}{4} \left( \frac{\partial V}{\partial u^g} - 2 L^T \vec{P} \right) - \frac{1}{4 \omega} \frac{d \omega}{dE} \frac{d \vec{u}^g}{dE}
\]

\[
\frac{dt}{dE} = \frac{1}{8 \omega} (K^2 - 2rV) - \frac{r}{16 \omega} \left( \vec{u}^g, \frac{\partial V}{\partial u^g} - 2 L^T \vec{P} \right) - \frac{2}{\omega} \frac{d \omega}{dE} \left( \vec{u}^g, \frac{d \vec{u}^g}{dE} \right)
\]

**The equations of motion (12) are now solved by the method of variation of constants. Assume that the solution has the form**

\[
\vec{u}^g = \vec{a}^g (E) \cos \frac{E}{2} + \vec{b}^g (E) \sin \frac{E}{2}
\]

**where the vectors \( \vec{a}^g \) and \( \vec{b}^g \) are the functions of the generalized eccentric anomaly. Differentiate with respect to \( E \), we get**

\[
\frac{d \vec{u}^g}{dE} = -\frac{1}{2} \vec{a}^g (E) \sin \frac{E}{2} + \frac{1}{2} \vec{b}^g (E) \cos \frac{E}{2} + \left( \vec{a}^g E \cos E \frac{d \vec{E}}{dE} + \frac{d \vec{b}^g}{dE} \sin E \right)
\]

**When \( \vec{a}^g \) and \( \vec{b}^g \) are constants, the terms inside the parenthesis vanishes, hence**

\[
\frac{d \vec{u}^g}{dE} = -\frac{1}{2} \vec{a}^g (E) \sin \frac{E}{2} + \frac{1}{2} \vec{b}^g (E) \cos \frac{E}{2}
\]

**This technique leads to a system of differential equations for \( \vec{a}^g (s) \) and \( \vec{b}^g (s) \), that is**

\[
\frac{d \vec{a}^g}{dE} = \left[ \frac{1}{2} \left( \frac{V}{2} \vec{u}^g + \frac{r}{4} \left( \frac{\partial V}{\partial u^g} - 2 L^T \vec{P} \right) \right) \right] + \frac{2}{\omega} \frac{d \omega}{dE} \frac{d \vec{u}^g}{dE} \sin E \]
When no perturbation occurs ($V = 0, \mathbf{P} = 0$), it can be seen that $\mathbf{\alpha}^\ast$ and $\mathbf{\beta}^\ast$ are constants. Together with these elements $\omega$ and $\tau$ they form a set of ten scalar elements and this is in agreement with the total order ten of the original differential system. In contrast to the classical elements these elements are well defined for any pure Kepler motion, even if collision occurs. Therefore, they are called regular elements. The differential equations describing the variation of the elements caused by the perturbations are the equations (13), (14) and (18). They are called as KS element equations of motion.

For convenience, we adopt the notation

$$\mathbf{u} = \mathbf{\beta}$$

Hence,

$$\mathbf{\alpha} = \frac{1}{2} \mathbf{\alpha}^\ast (E) \sin E + \frac{1}{2} \mathbf{\beta}^\ast (E) \cos E$$

The components of the position vector $\mathbf{x}^\ast$ of the particle are computed as

$$x_1 = u_1^2 - u_2^2 - u_3^2 + u_4^2,$$
$$x_2 = 2 (u_1 u_2 - u_3 u_4),$$
$$x_3 = 2 (u_1 u_3 - u_2 u_4)$$

The radial distance of the particle

$$r = u_1^2 + u_2^2 + u_3^2 + u_4^2$$

Velocity vector of the particle

$$\dot{x}_1 = \frac{4 \omega}{r} (u_1 u_1^* - u_2 u_2^* - u_3 u_3^* + u_4 u_4^*)$$
$$\dot{x}_2 = \frac{4 \omega}{r} (u_2 u_1^* + u_1 u_2^* - u_3 u_4^* + u_4 u_3^*)$$
$$\dot{x}_3 = \frac{4 \omega}{r} (u_3 u_1^* + u_4 u_2^* + u_1 u_3^* + u_2 u_4^*)$$

When, we assume the forces acting on an artificial satellite are those due to the Earth’s flattening (zonal and tesseral harmonics) only, thus we have

$$V = \frac{K^2}{r} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) \left( C_{n,m} \cos (m \lambda) + S_{n,m} \sin (m \lambda) \right) P_{n,m} \sin (\phi)$$

Also $\frac{dV}{dt} = - \omega_c \frac{dV}{\partial \lambda}, \omega_c = 7.292158 \times 10^{-5}$ radians per second is the rotational rate of the Earth. $R$ is mean equatorial radius of Earth, $\phi$ is the geocentric latitude, $\lambda$ is the longitude, $C_{nm}$ and $S_{nm}$ are dimensionless constants known as gravity coefficients for zonal, sectorial and tesseral harmonics, which can be computed using the following recurrence relations

$$P_{n,m} (x) = \frac{(2n-1) x P_{n-1,m} (x) - (n-1) P_{n+1,m} (x)}{n}; \quad m = 0$$
$$P_{n,m} (x) = (2n-1) \sqrt{1-x} P_{n-1,n-1} (x); \quad n = m$$
$$P_{n,m} (x) = \frac{(2n-1) x P_{n-1,m} (x) - (m+n-1) P_{n-2,m} (x)}{n-m}; \quad m \neq n$$
$$P_{m,0} (x) = 1, \quad P_{1,0} (x) = x, \quad P_{1,1} (x) = - \sqrt{1-x^2}$$

**Derived Legendre Function and Normalized Geo-potential Coefficients**

The expression given in (24) results in some computational difficulties. One of the difficulties is the range of the magnitudes of the parameters $P_{n,m}, C_{nm}$ and $S_{nm}$ as $n$ increases. So to avoid the large variations in the magnitudes of these parameters normalized Legendre functions have been proposed, which have a more graceful change in the exponent.

The normalizing factor is defined such that the normalized spherical harmonics will have mean square value of one on the unit sphere. The normalized Legendre functions are defined such that the product of the gravity coefficients and the corresponding Legendre functions remain constant [8, 9].
i.e. \( P_{n,m} C_{n,m} = \overline{P}_{n,m} \overline{C}_{n,m} \) and \( P_{n,m} S_{n,m} = \overline{P}_{n,m} \overline{S}_{n,m} \),

where \( P_{n,m} \overline{C}_{n,m} \) and \( S_{n,m} \overline{C}_{n,m} \) are the normalized functions for geo-potential coefficients. The typical normalizing factor is taken to be

\[
N_{n,m} = \sqrt{(n-1)! (2n+1) (2 - \partial_{0,m}) / (n + m)!}
\]

and thus

\[
P_{n,m} = N_{n,m} P_{n,m}, \quad \overline{C}_{n,m} = C_{n,m} / N_{n,m}
\]

\[
\overline{S}_{n,m} = S_{n,m} / N_{n,m},
\]

where \( \partial_{0,m} \) is the kronecker delta functions which is defined as

\[
\partial_{0,m} = \begin{cases} 1, & m = 0 \\ 0, & \text{otherwise} \end{cases}
\]

**Numerical Results**

The numerical integration of the above differential equations of motion with Earth’s zonal and tesseral harmonics up to \( J_{12,12} \) (whose values are taken from WGS84_EGM96 model [9]) has been carried out with a fixed step size fourth order Range-Kutta method with respect to the initial conditions provided in Table-1. These initial conditions, correspond to the epoch of 13th July 1988 00:00:00 UTC of an actual artificial satellite IRS-1A with NORAD number 18960, which was inserted into a near-circular orbit on 17th March 1988. The constants used for Earth radius (R) and Gravitational constant \((k^2)\) are 6378.145 km and 398600.4418 km²/s² respectively. Table-2 consists of the osculating orbital elements including perigee altitude (Hp) and apogee altitude (Ha) obtained with a step size of 1 degree in eccentric anomaly using the present KS theory with only Earth’s zonal harmonics terms (Z) up to \( J_{12} \) and Earth’s flattening terms (T) up to \( J_{12,12} \) for IRS-1A satellite orbit, which is compared with actual observed values (O) for 30 days. It is noticed that the percentage error \([100(Observed-Predicted)/Observed]\) for the important orbital parameter semi-major axis, which is a measure of energy is only 0.001 with the inclusion of tesseral harmonics, where as it is doubled (0.002) with zonal harmonic alone. Similar trend is noticed for other parameters also. Figs.1 to 6 shows the differences between the actual and predicted values of the orbital parameters semi-major axis, eccentricity, inclination and right ascension of ascending node as well as perigee height and apogee height with Earth’s zonal harmonics terms (Z) up to \( J_{12} \) and Earth’s flattening terms (T) up to \( J_{12,12} \) for IRS-1A satellite. It is evident from the Table-2 and Figs.1 to 6 that the maximum difference between the observed and predicted values with respect to zonal harmonics and flattening (tesseral harmonics) for the orbital parameters semi-major axis, eccentricity, inclination, right ascension of ascending node, perigee and apogee altitudes are 113.9, 84.6 meters; 0.000052, 0.000038; 0.0041, 0.0031 degrees; 0.011, 0.004 degrees; 426.6, 279.9 meters 219.8, 213.6 meters respectively for 30 days. As expected, inclusion of tesseral harmonics improved the predictions. Predicted result shows that the KS theory can be utilized for accurate orbit prediction.

To know the effectiveness of the present theory (KS), the results were also compared with High Precision Orbit Propagator (HPOP) available in System Tool Kit (STK 9.2.1). For both KS and HPOP theories, the equations of motions are integrated with Runge-Kutta fourth order method by including Earth’s gravity terms (zonal, sectorial and tesseral terms) up to \( J_{66} \), as the perturbing force Figs.7 and 8 depicts the difference between actuals and predicted values with respect to KS and HPOP theories for

<table>
<thead>
<tr>
<th>Table-1: Initial Conditions (IRS-1A Satellite)</th>
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<tr>
<td>Position and Velocity</td>
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<tr>
<td>(x_1) (km)</td>
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<td>(x_2) (km)</td>
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<td>(x_3) (km)</td>
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<td>(\dot{x}_1) (km/s)</td>
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the important orbital parameters semi-major axis and eccentricity, which describe the size and shape of the orbit. From these Figs. 7 and 8, it is clearly observed that KS theory is better than HPOP theory.

Conclusion

This paper provides the details of the study carried out on KS element equations with Earth’s gravity harmonics with zonal and tesseral terms for long term orbit computation. The effectiveness of the method is illustrated by comparing the KS theory with the actual IRS-1A satellite data and also with HPOP results. The study reveals that KS element equations with force model Earth’s flattening as perturbation provides an efficient and accurate method for orbit prediction even for a long duration. Any number of zonal and tesseral harmonic terms can be included using the recurrence formula for Legendre polynomials, its derivatives and associate Legendre polynomials, which economizes the computer time and effort.

Acknowledgements

Authors acknowledges the Editor as well as Reviewers of the paper for their good comments, which enhanced the quality of the paper.

References


Fig.1 Difference Between Actual and Predicted Values of Semi-major Axis

Fig.2 Difference Between Actual and Predicted Values of Eccentricity
Fig. 3 Difference Between Actual and Predicted Values of Inclination

Fig. 4 Difference Between Actual and Predicted Values of Right Ascension of Ascending Node

Fig. 5 Difference Between Actual and Predicted Values of Perigee

Fig. 6 Difference Between Actual and Predicted Values of Apogee

Fig. 7 Difference Between KS and HPOP Values of Semi-major Axis

Fig. 8 Difference Between KS and HPOP Values of Eccentricity