BENDING OF FUNCTIONALLY GRADED SPHERICAL SHELLS

K.S. Sai Ram; B. Swathi; K. P. Manjusha; K. Srinivasu
Department of Civil Engineering
RVR & JC College of Engineering
Chowdavaram, Guntur-522 019, India
Email: sairamks@yahoo.com

Abstract

The bending behavior of functionally graded spherical shells with a cutout is studied using the finite element method based on a higher-order shear deformation theory. The higher-order shear deformation theory is derived by assuming that the transverse displacement is constant through the thickness of the shell. Material properties are assumed to be graded in the thickness direction according to a simple power law distribution in terms of the volume fractions of the constituents. An eight-noded degenerated isoparametric shell element is considered with nine degrees of freedom at each node. The element stiffness matrix and load vector are derived using the principle of minimum potential energy. The formulation and the programme code are validated by comparing the results with those available in the literature. Results are presented for the variation of deflection and stresses in functionally graded spherical shell cap with a circular cutout with simply supported and clamped boundary conditions subjected to uniform normal pressure.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>a, b</td>
<td>Arc-lengths of the shell in XZ and YZ planes, respectively</td>
</tr>
<tr>
<td>E_c, E_m</td>
<td>Young’s moduli of ceramic and metal, respectively</td>
</tr>
<tr>
<td>K_x, K_y, K_xy</td>
<td>Curvatures of a shell</td>
</tr>
<tr>
<td>M_x, M_y, M_xy</td>
<td>Moment resultants per unit length</td>
</tr>
<tr>
<td>N_x, N_y, N_xy</td>
<td>Membrane forces per unit length</td>
</tr>
<tr>
<td>n</td>
<td>Volume fraction exponent</td>
</tr>
<tr>
<td>p_0</td>
<td>Intensity of normal pressure</td>
</tr>
<tr>
<td>Q_x, Q_y</td>
<td>Transverse shear forces per unit length</td>
</tr>
<tr>
<td>R_x, R_y</td>
<td>Radii of curvature in XZ and YZ planes, respectively</td>
</tr>
<tr>
<td>t</td>
<td>Thickness of a shell</td>
</tr>
<tr>
<td>u, v, w</td>
<td>Displacement components along x, y and z axes, respectively</td>
</tr>
<tr>
<td>u_0, v_0</td>
<td>Displacements of the mid-surface along x and y axes, respectively at a node</td>
</tr>
<tr>
<td>W_0</td>
<td>= Deflection and normalised deflection along Z-axis, respectively</td>
</tr>
<tr>
<td>U_0i, V_0i, W_0i</td>
<td>= Displacements of the mid-surface along X, Y and Z axes, respectively at a node i</td>
</tr>
<tr>
<td>u_0,x, v_0,x, w_0,x</td>
<td>= Derivatives of a variable with respect to a subscript</td>
</tr>
<tr>
<td>x, y, z</td>
<td>= Local Cartesian coordinate axes at any point on the mid-surface of a shell, x and y axes being tangential to the mid-surface whereas z-axis is normal the mid-surface</td>
</tr>
<tr>
<td>X_i, Y_i, Z_i</td>
<td>= Global cartesian coordinates of a node i</td>
</tr>
<tr>
<td>γ_x, γ_y</td>
<td>= Strains along x and y axes, respectively</td>
</tr>
<tr>
<td>γ_x0, γ_y0</td>
<td>= Strains of the mid-surface along x and y axes, respectively</td>
</tr>
<tr>
<td>γ_xy</td>
<td>= Shear strain in xy plane at a distance z from the mid-surface</td>
</tr>
<tr>
<td>γ_xy0</td>
<td>= Shear strain of the mid-surface in xy plane</td>
</tr>
<tr>
<td>γ_xz, γ_yz</td>
<td>= Transverse shear strains at a distance z from the mid-surface</td>
</tr>
<tr>
<td>ε_x, ε_y</td>
<td>= Strains along x and y axes, respectively at a distance z from the mid-surface</td>
</tr>
<tr>
<td>ε_x0, ε_y0</td>
<td>= Strains of the mid-surface along x and y axes, respectively</td>
</tr>
</tbody>
</table>
Functionally Graded Materials (FGMs) are receiving increased attention from engineers since the properties of these materials can be continuously varied through their volume to suit the working environments. These materials have potential application in aerospace engineering. The most common FGMs are metal / ceramic composites, which the ceramic part has good thermal resistance and the metallic part has superior fracture toughness. FGMs, in the form of shells, may be used in various applications. Functionally graded spherical shell caps are used in various applications such as aerospace vehicles, missiles and pressure vessels. In actual application, these shell caps may have a cutout. The presence of a cutout produces stress concentration and reduces the strength and stiffness of functionally graded spherical shell caps.

Most of the earlier researchers considered the bending, vibration and buckling of functionally graded cylindrical shells [1-21]. The research conducted in the area of structural behavior of functionally graded spherical shells is meager. Ganapathi [22] studied the dynamic stability of clamped functionally graded spherical shell subjected to external pressure. Numerical results are presented showing the effects of power-law index of functionally graded material on the axisymmetric dynamic stability characteristics of shallow spherical shells. Bich and Tung [23] investigated the nonlinear behaviour of spherical shell caps subjected to external pressure with and without the effects of temperature using Galerkin method based on classical shell theory. Bich et al. [24] presented non-linear static and dynamic buckling analysis of functionally graded spherical caps subjected to external pressure and temperature using Galerkin method based on classical shell theory. Fadaee [25] developed a closed-form analysis for natural vibrations of functionally graded thick spherical shell panel. The effects of various stretching-bending couplings on the frequency parameters are discussed. Alashti et al. [26] investigated the thermo-elastic analysis of functionally graded spherical shell with piezoelectric layers under the effect of thermo-electro-mechanical loading. The effects of the grading index of material properties, temperature difference and thickness of piezoelectric layers on stress, displacement and temperature fields are presented. Ye et al. [27] studied the free vibration of laminated functionally graded spherical shells. Parametric studies are carried out to examine the influences of boundary conditions, geometric parameters and material distributions on the natural frequencies of spherical shells. Khaire and Jagtap [28] investigated nonlinear free vibration response of functionally graded spherical shell subjected to thermomechanical loading. Kar and Panda [29] presented nonlinear finite element analysis of functionally graded spherical shell panels. The variation of deflection of spherical shell panels with load has been studied.

From the above review of literature, it is evident that the most of the researchers considered functionally graded cylindrical shells. A few researchers considered geometric non-linear analysis, vibration and buckling of spherical shells. To the best of authors’ knowledge, no work has been done on the bending behavior of functionally graded spherical shells with a cutout. Hence, in this paper, the bending behaviour of functionally graded spherical shell cap with a circular cutout is studied using the finite element method based on a higher-order shear deformation theory. A higher-order theory is used to properly account for transverse shear deformation. The higher-order theory is derived by assuming that the transverse displacement is constant through the thickness of the shell. An eight noded degenerated isoparametric shell element is used.
with nine degrees of freedom at each node. Results are
presented for the variation of deflection and stresses in
functionally graded spherical shell cut with a circular
cutout with simply supported and clamped boundary con-
ditions subjected to uniform normal pressure.

Governing Equations
Consider a functionally graded shell panel of uniform
thickness as shown in Fig.1. The displacements along the
local coordinate axes x, y and z at any point in the shell
are assumed as

\[ u = u_0 + z \theta + z^2 u_0^* + z^3 \theta^* \]
\[ v = v_0 - z \theta + z^2 v_0^* - z^3 \theta^* \]
\[ w = w_0 \]  

(1)

The strains along the local coordinate axes x, y and z
are given by

\[ \varepsilon_x = \varepsilon_{x0} + z K_x + z^2 \varepsilon_{x0} + z^3 K_x^* \]
\[ \varepsilon_y = \varepsilon_{y0} + z K_y + z^2 \varepsilon_{y0} + z^3 K_y^* \]
\[ \gamma_{xy} = \gamma_{x0} + z K_{xy} + z^2 \gamma_{x0} + z^3 K_{xy}^* \]
\[ \gamma_{xz} = \gamma_{x0} + z w + \phi_x + z^2 \gamma_{x0}^* + z^3 \phi_x^* \]
\[ \gamma_{yz} = \gamma_{y0} + z w + \phi_y + z^2 \gamma_{y0}^* + z^3 \phi_y^* \]  

(2)

where

\[ \varepsilon_{x0} = u_{0x}, \varepsilon_{y0} = u_{0y}, \gamma_{x0} = u_{0x, y}, \gamma_{y0} = u_{0y, x} \]
\[ K_x = \theta_{y, x}, K_y = -\theta_{x, y}, K_{xy} = \theta_{x, y} \]
\[ \varepsilon_{x0}^* = u_{0x}, \phi_x^* = v_{0x, y}, \gamma_{x0}^* = u_{0x, y} + v_{0x, y} \]
\[ K_x^* = \theta_{y, x}^*, K_y^* = -\theta_{x, y}^*, K_{xy}^* = \theta_{x, y}^* \]
\[ \phi_x = \theta_x + w_{0x}, \phi_y = -\theta_y + w_{0y}, \phi_x^* = \theta_x^* + z \phi_x^* \]
\[ \gamma_{x0}^* = 2 w_{0x}, \gamma_{xy0}^* = 2 v_{0x} \]

(3)

The stress resultants are given by

\[ \sigma_x = Q_{11} \varepsilon_x + Q_{12} \varepsilon_y + \gamma_x \]
\[ \sigma_y = Q_{12} \varepsilon_x + Q_{22} \varepsilon_y + \gamma_y \]
\[ \tau_{xy} = Q_{66} \gamma_{xy} \]

(4a)

\[ \tau_{xz} = Q_{44} \varepsilon_x + Q_{55} \varepsilon_y + \gamma_x \]
\[ \tau_{yz} = Q_{45} \varepsilon_x + Q_{54} \varepsilon_y + \gamma_y \]

(4b)

where

\[ Q_{11} = Q_{22} = E (z)/(1 - v^2) \]
\[ Q_{12} = v Q_{11} \]
\[ Q_{44} = Q_{55} = Q_{66} = E (z)/2 (1 - v) \]
\[ E (z) = (E_c - E_m) V_c + E_m \]
\[ V_c = (z/h + 1/2)^n \]

The stress resultants are given by

\[ \begin{bmatrix} N_x & M_x & N_y & M_y \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \tau_x \\ \tau_y \\ \tau_{xy} \end{bmatrix} [1 - z^2 - z^3] dz \]

(5a)
\[
\begin{bmatrix}
Q_x & S_x & Q_x^* \\
Q_y & S_y & Q_y^*
\end{bmatrix} = \int_{-l/2}^{l/2} \begin{bmatrix}
\tau_{xz}
\\
\tau_{yz}
\end{bmatrix} \begin{bmatrix}
1 & z & z^2
\end{bmatrix}
\]

(5b)

The constitutive relationship for the shell may be written as

\[
| F | = [D] \begin{bmatrix}
\chi
\end{bmatrix}
\]

(6)

\[
| F | = [N]_{x,y}, [M]_{x,y}, [N]_{xy}, [N]_{x}^{*}, [N]_{y}^{*}, [N]_{xy}^{*}
\]

\[
M_{x}^{*}, M_{y}^{*}, M_{xy}^{*}, Q_{x}, Q_{y}, S_{x}, S_{y}, Q_{x}^{*}, Q_{y}^{*}
\]

\[
\begin{bmatrix}
\chi
\end{bmatrix} = \begin{bmatrix}
\varepsilon_{x0}, \varepsilon_{y0}, \gamma_{x0y}, K_{x}, K_{y}, K_{xy}, \varepsilon_{x0}^{*}, \varepsilon_{y0}^{*}, \gamma_{x0y}^{*}, K_{x}^{*}, K_{y}^{*}, K_{xy}^{*}, \phi_{x}, \phi_{y}, \gamma_{xyc}, \phi_{x}^{*}, \phi_{y}^{*}
\end{bmatrix}^{T}
\]

The elasticity matrix [D] in Eqn.(6) may be expressed as

\[
[D] = \begin{bmatrix}
[A_{ij}] & [B_{ij}] & [C_{ij}] & [D_{ij}] & [0] & [0] & [0] \\
[B_{ij}] & [C_{ij}] & [D_{ij}] & [E_{ij}] & [0] & [0] & [0] \\
[C_{ij}] & [D_{ij}] & [E_{ij}] & [F_{ij}] & [0] & [0] & [0] \\
[D_{ij}] & [E_{ij}] & [F_{ij}] & [G_{ij}] & [0] & [0] & [0] \\
[0] & [0] & [0] & [0] & [A_{pq}] & [B_{pq}] & [C_{pq}] \\
[0] & [0] & [0] & [0] & [B_{pq}] & [C_{pq}] & [D_{pq}] \\
[0] & [0] & [0] & [0] & [C_{pq}] & [D_{pq}] & [E_{pq}]
\end{bmatrix}
\]

The validity of the present finite element analysis is further validated by comparing the results with those of other researchers. The element stiffness matrices and element load vectors are assembled to obtain the respective global matrices [K] and [P]. The unknown displacements at the nodes of the shell [\delta] are obtained from the equilibrium condition.

\[
[K] \begin{bmatrix}
\delta
\end{bmatrix} = [P]
\]

(7)

This is solved using the Gauss elimination technique [33]. Knowing the displacements at the nodes, strains and then stresses are evaluated.

Results and Discussion

The analysis described in the previous sections is applicable for the bending of various types of functionally graded shells subjected to uniform normal pressure. In the present investigation, results are presented for functionally graded spherical shell cap with a circular cutout (\(\phi_0 = 2^\circ, \phi_1 = 10^\circ\)) with simply supported and clamped boundary conditions subjected to uniform normal pressure. In the case of simply supported functionally graded spherical shell cap with a circular cutout (Fig.2), V_{0i} and U_{0i} are restrained along the supported edge. For functionally graded spherical shell cap, the local Cartesian coordinate axes x and y are oriented such that x-axis is in the circumferential direction and y-axis is in the meridional direction. The following properties are used in the investigation.

\[E_{c} = 151 \text{ GPa}, \ E_{m} = 70 \text{ GPa}, \ \nu = 0.3\]

To study the convergence of results, the entire spherical shell cap is discretised with 80, 96 and 112 elements. It is found that convergence is obtained with 112 elements (Fig.2) and hence it is used in the present investigation. The validation of the present finite element analysis is carried out in the following cases.

- A simply supported square plate subjected to uniformly distributed normal load (Table-1).
- A simply supported square cylindrical shell panel subjected to uniformly distributed normal load (Table-2, Fig.3).
From the results in Tables-1, 2 and Fig.3, it is clear that the present finite element analysis is accurate and reliable.

Figures 4-5 show the variation of normalised deflection $W = 10^6 E_c W/p_0 (R/t)^4$ along the meridian of functionally graded spherical shell cap with a circular cutout for simply supported and clamped boundary conditions for radius to thickness ratios 500 and 50.

Figures 6-7 show the variation of normalised normal stresses at the top and the bottom $\sigma_{x} = 10^3 \sigma_{x}/p_0 (R/t)^2$, $\sigma_{y} = 10^3 \sigma_{y}/p_0 (R/t)^2$, $\tau_{xy} = 10^3 \tau_{xy}/p_0 (R/t)^2$, $\tau_{xz} = 10^3 \tau_{xz}/p_0 (R/t)^2$, and $\tau_{yz} = 10^3 \tau_{yz}/p_0 (R/t)^2$ along the meridian of functionally graded spherical shell cap with a circular cutout for simply supported boundary condition for radius to thickness ratios 500 and 50.

Figures 8-9 show the variation of normalised normal stresses at the top and the bottom $\sigma_{x} = 10^3 \sigma_{x}/p_0 (R/t)^2$, $\sigma_{y} = 10^3 \sigma_{y}/p_0 (R/t)^2$, $\tau_{xy} = 10^3 \tau_{xy}/p_0 (R/t)^2$, $\tau_{xz} = 10^3 \tau_{xz}/p_0 (R/t)^2$, and $\tau_{yz} = 10^3 \tau_{yz}/p_0 (R/t)^2$ along the meridian of functionally graded spherical shell cap with a circular cutout for simply supported boundary condition for radius to thickness ratios 500 and 50.

Figures 10-11 show the variation of normalised normal stresses at the top and the bottom $\sigma_{x} = 10^3 \sigma_{x}/p_0 (R/t)^2$, $\sigma_{y} = 10^3 \sigma_{y}/p_0 (R/t)^2$, $\tau_{xy} = 10^3 \tau_{xy}/p_0 (R/t)^2$, $\tau_{xz} = 10^3 \tau_{xz}/p_0 (R/t)^2$, and $\tau_{yz} = 10^3 \tau_{yz}/p_0 (R/t)^2$ along the meridian of functionally graded spherical shell cap with a circular cutout for clamped boundary condition for radius to thickness ratios 500 and 50.

Figures 12-13 show the variation of normalised normal stresses at the top and the bottom $\sigma_{x} = 10^3 \sigma_{x}/p_0 (R/t)^2$, $\sigma_{y} = 10^3 \sigma_{y}/p_0 (R/t)^2$, $\tau_{xy} = 10^3 \tau_{xy}/p_0 (R/t)^2$, $\tau_{xz} = 10^3 \tau_{xz}/p_0 (R/t)^2$, and $\tau_{yz} = 10^3 \tau_{yz}/p_0 (R/t)^2$ along the meridian of functionally graded spherical shell cap with a circular cutout for clamped boundary condition for radius to thickness ratios 500 and 50.
The deflection of a spherical shell cap with a circular cutout is compressive except near the support where it is tensile (Fig.10). It is maximum near the hole edge for R/t ratios 500 and 50. The normal stress in the circumferential direction at the bottom in a clamped spherical shell cap with a circular cutout is compressive throughout the cap and is maximum at the hole edge for R/t ratio 500 (Fig.11(b)). This is due to curvature of the shell, rigidity at the support and thickness effect.

The normal stress in the meridional direction at the top in a clamped spherical shell cap with a circular cutout is compressive throughout the spherical shell cap except near the support where it is in tensile for R/t ratio 500 and it is maximum at φ which is about 7.5° for R/t=500 (Fig.12(a)). It is compressive up to φ which is about 7° for R/t ratio 50 and becomes tensile in the remaining part, maximum being on the support (Fig.12(b)). The normal stress in the meridional direction at the bottom in a clamped spherical cap with a circular cutout is compressive throughout the cap and is maximum on the support for R/t ratio 500 (Fig.13(a)); it is tensile near the hole edge and compressive in the remaining part, maximum being on the support for R/t ratio 50 (Fig.13(b)). This is due to curvature of the shell, rigidity at the support and thickness effect.

Conclusions

In this investigation, the bending behavior of functionally graded spherical shells is studied using finite element method based on a higher-order shear deformation theory. An eight noded degenerated isoparametric shell element with nine degrees of freedom at each node is considered. Results are presented for the variation of deflection and stresses in the spherical shell cap with a circular cutout with simply supported and clamped boundary conditions subjected to uniform normal pressure.

- The deflection of a spherical shell cap with a circular cutout may be maximum at the hole edge or away from the hole edge depending on radius to thickness ratio.
- The normal stress at the top in a simply supported spherical cap with a circular cutout is compressive except near the support where it is tensile (Fig.10). It is maximum near the hole edge for R/t ratios 500 and 50. The normal stress in the circumferential direction at the bottom in a clamped spherical shell cap with a circular cutout is compressive throughout the cap and is maximum at the hole edge for R/t ratio 500 (Fig.11(b)). This is due to curvature of the shell, rigidity at the support and thickness effect.
• The normal stress at the bottom in the circumferential direction in simply supported and clamped spherical shell caps may be compressive or tensile depending on radius to thickness ratio.

• The normal stress at the top in the meridional direction in a simply supported shell cap is always compressive and it is maximum at $\phi$ which varies from 5° to 8.25° depending on $R/t$ ratio. It changes from compression to tension towards the support in a clamped spherical shell cap.

• The normal stress at the bottom in the meridional direction in simply supported and clamped spherical shell caps is predominantly compressive, maximum being on the support.

References


Fig. 1 Functionally Graded Shell Panel (a) Geometry

Fig. 1 Functionally Graded Shell Panel (b) Material Variation Through Thickness

Fig. 2 Functionally Graded Spherical Shell Cap with a Circular Cutout (a) Geometry

Fig. 2 Functionally Graded Spherical Shell Cap with a Circular Cutout (b) Discretisational Details for a Quarter

Fig. 3 Comparison of Axial Stress at the Centre of Simply Supported Functionally Graded Cylindrical Shell Panel (a) From Present Analysis

Fig. 3 Comparison of Axial Stress at the Centre of Simply Supported Functionally Graded Cylindrical Shell Panel (b) From Reference [18]
Fig. 4 Variation of Deflection of a Simply Supported Spherical Shell Cap with a Circular Cutout Along the Meridian $(\phi_0 = 2^\circ, \phi_1 = 10^\circ)$ (a) $R/t = 500$

Fig. 5 Variation of Deflection of a Clamped Spherical Shell Cap with a Circular Cutout Along the Meridian (a) $R/t = 500$

Fig. 6 Variation of Normal Stress $\sigma_x$ Along the Meridian of a Simply Supported Spherical Shell Cap with a Circular Cutout (a) $R/t = 500$

Fig. 4 Variation of Deflection of a Simply Supported Spherical Shell Cap with a Circular Cutout Along the Meridian $(\phi_0 = 2^\circ, \phi_1 = 10^\circ)$ (b) $R/t = 50$

Fig. 5 Variation of Deflection of a Clamped Spherical Shell Cap with a Circular Cutout Along the Meridian (b) $R/t = 50$

Fig. 6 Variation of Normal Stress $\sigma_x$ Along the Meridian of a Simply Supported Spherical Shell Cap with a Circular Cutout (b) $R/t = 50$
Fig. 7 Variation of Normal Stress $\sigma_x^b$ Along the Meridian of a Simply Supported Spherical Shell Cap with a Circular Cutout
(a) $R/t = 500$

Fig. 7 Variation of Normal Stress $\sigma_x^b$ Along the Meridian of a Simply Supported Spherical Shell Cap with a Circular Cutout
(b) $R/t = 50$

Fig. 8 Variation of Normal Stress $\sigma_y^t$ Along the Meridian of a Simply Supported Spherical Shell Cap with a Circular Cutout
(a) $R/t = 500$

Fig. 8 Variation of Normal Stress $\sigma_y^t$ Along the Meridian of a Simply Supported Spherical Shell Cap with a Circular Cutout
(b) $R/t = 50$

Fig. 9 Variation of Normal Stress $\sigma_y^b$ Along the Meridian of a Simply Supported Spherical Shell Cap with a Circular Cutout
(a) $R/t = 500$

Fig. 9 Variation of Normal Stress $\sigma_y^b$ Along the Meridian of a Simply Supported Spherical Shell Cap with a Circular Cutout
(b) $R/t = 50$
Fig. 10 Variation of Normal Stress $\sigma_x$ along the Meridian of a Clamped Spherical Shell Cap with a Circular Cutout
(a) $R/t = 500$

Fig. 10 Variation of Normal Stress $\sigma_x$ along the Meridian of a Clamped Spherical Shell Cap with a Circular Cutout
(b) $R/t = 50$

Fig. 11 Variation of Normal Stress $\sigma_y$ along the Meridian of a Clamped Spherical Shell Cap with a Circular Cutout
(a) $R/t = 500$

Fig. 11 Variation of Normal Stress $\sigma_y$ along the Meridian of a Clamped Spherical Shell Cap with a Circular Cutout
(b) $R/t = 50$

Fig. 12 Variation of Normal Stress $\sigma_z$ along the Meridian of a Clamped Spherical Shell Cap with a Circular Cutout
(a) $R/t = 500$

Fig. 12 Variation of Normal Stress $\sigma_z$ along the Meridian of a Clamped Spherical Shell Cap with a Circular Cutout
(b) $R/t = 50$
Fig. 13 Variation of Normal Stress $\bar{\sigma}_b^b$ Along the Meridian of a Clamped Spherical Shell Cap with a Circular Cutout
(a) $R/t = 500$

Fig. 13 Variation of Normal Stress $\bar{\sigma}_b^b$ Along the Meridian of a Clamped Spherical Shell Cap with a Circular Cutout
(b) $R/t = 50$