A bifurcation theoretic integrated methodology for aircraft conceptual design is presented in this paper. The methodology uses the extended bifurcation analysis technique to formulate a unified design framework that can be applied for configuration sizing of aircraft, assessment of aircraft performance and stability, and evaluation of aircraft maneuverability. Unlike conventional approaches for conceptual design, the proposed methodology incorporates a full six degrees-of-freedom flight dynamics model for stability and control analysis. Maneuverability evaluation is done with the help of a suitable controller. Configuration sizing of a six-seater general aviation aircraft, followed by performance, stability and maneuverability assessment of the aircraft is successfully carried out in this paper to demonstrate the effectiveness of the proposed methodology.

**Keywords:** Aircraft design; Bifurcation analysis; Stability and control; Performance and maneuverability

**Nomenclature**

- $V$ = Velocity of aircraft, ft/s
- $\alpha, \beta$ = Angle of attack and sideslip angle, respectively, rad
- $p, q, r$ = Body axis roll, pitch and yaw rates, respectively, rad/s
- $\phi, \theta, \psi$ = Body axis Euler roll, pitch and yaw angles, respectively, rad
- $\mu, \gamma$ = Wind axis orientation angles, rad
- $\delta e, \delta a, \delta r$ = Elevator, aileron and rudder deflections, respectively, rad
- $C_L, C_D, C_Y$ = Lift, drag and side force coefficient, respectively
- $C_I, C_m, C_n$ = Aerodynamic rolling, pitching and yawing moment coefficient, respectively
- $T_m$ = Maximum available engine thrust, lb
- $\eta_T$ = Thrust as a fraction of maximum available engine thrust
- $m$ = Mass of aircraft, slug
- $X_{cg}$ = Location of center-of-gravity of aircraft, as percentage of mean aerodynamic chord
- $\omega_n$ = Undamped natural frequency, rad/s
- $g$ = Acceleration due to gravity, ft/s$^2$
- $\zeta$ = Damping ratio
- $t_1/2, t_2$ = Time to halve amplitude and time to double amplitude, seconds

**Introduction**

Conceptual design is one of the most important stages in the development of a new aircraft. The basic aircraft structure with its main components is first defined during conceptual design phase. The main design activities that are accomplished during conceptual design phase are: weight and balance computation; selection of airfoil; calculation of aspect ratio and planform area of wing; choice of powerplant; definition of fuselage configuration; sizing of control surfaces; choice of landing gear; and analysis of the aircraft performance and stability. The design process...
in conceptual phase is mainly based on historical data and simple analytical methods [1, 2]. With rapid availability of design softwares, conceptual design is generally carried out using software packages that essentially implement analytical methods and empirical relations for sizing of aircraft. Increasingly, use of softwares that incorporate different optimization techniques (for e.g., RDS design software from Dan Raymer) has become popular during conceptual design stage. Design softwares are frequently used to arrive at configurations that are optimized to minimize or maximize one or more objective functions subject to various constraints on the vehicle performance. Shirley et al. [3] examined optimal designs that maximized aerodynamic performance and minimized structural weight for a medium-range transport aircraft. Optimal aircraft configurations for minimizing aircraft noise and emissions were investigated in [4, 5]. Studies have been carried out to obtain robust optimal designs that are insensitive to uncertainties present in simplified analytical models used for conceptual design process [6, 7]. Optimization tools that allow the designer to take into account aeroelasticity effects during conceptual design stage have been developed [8]. Multi-fidelity, multidisciplinary design optimization codes that provide a host of design options to the designer during various design phases to expedite the entire design cycle are being seriously pursued [9]. Optimization techniques have been applied for design of Unmanned Combat Air Vehicle (UCAV) as well [10].

Presently, design softwares with a plethora of features are readily available to the designer. Most of the softwares for conceptual design, however, do not give adequate importance to stability and control analysis and maneuverability assessment of an aircraft. Stability and control analysis is either postponed till later design stages or simplified linearized equations are used for evaluating static and dynamic stability of an aircraft. A conceptual design methodology that includes modules for a sufficiently detailed stability and control analysis, and maneuverability evaluation will greatly aid in development of an aircraft which will have better chances of meeting the desired handling qualities and maneuver characteristics. Such a design methodology will also be of significant help in development of UCAVs which are designed to be inherently unstable with a high degree of maneuverability.

A bifurcation analysis-based design methodology is presented in this paper. Bifurcation analysis technique has conventionally been used to examine nonlinear flight dynamics of aircraft [11, 12]. Trim and stability of nonlinear aircraft models under constrained flight conditions have also been studied using bifurcation analysis technique [13, 14, 15]. A general design framework for incorporating bifurcation and continuation methods within the aircraft systems design cycle was described in Ref. [16]. In this work, an integrated methodology for conceptual design of an aircraft is presented. The methodology uses a full 6 degrees-of-freedom (DOF) flight dynamics model instead of simplified linearized models or performance metrics for a detailed investigation of the aircraft performance, stability and maneuverability. A unique feature of the suggested methodology is that it provides a unified design and analysis framework that can be employed for configuration sizing of an aircraft, detailed assessment of the aircraft performance and stability, and also for rigorous evaluation of the aircraft maneuverability by studying maneuver design problem.

This paper is organized as follows. A brief description of the extended bifurcation analysis method is given in Section-The Extended Bifurcation Analysis Method. Section-Aircraft Flight Dynamics Model, describes the 6-DOF flight dynamics model used here, and Section-Preliminary Design Data provides design data for the six-seater aircraft taken in this work. The bifurcation theoretic integrated design methodology is presented in the next Section; the methodology is later demonstrated by applying it for conceptual design of the six-seater aircraft. Finally, the Section-Conclusions, concludes the work presented in this paper.

The Extended Bifurcation Analysis Method

The Extended Bifurcation Analysis (EBA) method [14] is an algorithm for bifurcation analysis of a constrained nonlinear dynamical system. The EBA method consists of two steps. In step 1 of the EBA method, governing equations for the system are simultaneously solved with the constraint equations. Thus, solve:

\[ \dot{x} = f(x, u, p) = 0 \]
\[ g(x) = 0 \]  \hspace{1cm} (1)

where \( x \in X \subset \mathbb{R}^n \) is the vector of state variables, and \( u \in U \subset \mathbb{R} \) is the continuation parameter. The vector function \( g(x) \) represents a \( k \)-dimensional vector of constraints, and vector \( p \) denotes a \( m \)-dimensional bounded set of control parameters. It is necessary that the number of control parameters in \( p \) should at least be equal to the number of imposed constraints (i.e., \( m \geq k \)) for the constrained bifurcation analysis problem to be well-posed. To solve for the constrained system represented by Eqn.(1), a number of
control parameters (equal to the number of constraints) are freed from vector $p$ while remaining control parameters are kept fixed. It is further imperative that the free control parameters must influence the specified constraints. Output from this step gives schedules for the free control parameters required to satisfy the stipulated constraints. Additionally, steady state solutions satisfying the prescribed constraints are computed in this step; stability information of the computed steady state solutions is however incorrect.

Using the parameter schedules computed in step 1, the following set of equations is solved in step 2 of the EBA method:

$$\dot{x} = f(x, u, p_1(u), p_2) = 0$$  \hspace{1cm} (2)

where $p_1(u)$ are schedules for the free control parameters as obtained from step 1, and $p_2$ are the fixed control parameters. Output of this step provides steady state solutions with correct stability information. In addition, in this step, bifurcation solution branches emanating from the solution branch satisfying the constraints are also computed. The bifurcated solution branches indicate departure from constrained conditions. Thus, EBA approach results in computation of global dynamic behavior of a constrained nonlinear system. All the bifurcation analysis results for the aircraft model taken in this work are obtained through AUTO 2000 bifurcation and continuation software [17].

**Aircraft Flight Dynamics Model**

A full 6-DOF rigid body aircraft flight dynamics model is implemented within the design methodology for stability and control analysis. The 6-DOF flight dynamics model is generally represented by a set of first order, nonlinear, ordinary differential equations as provided in the Appendix. Flight dynamics model for a particular aircraft is characterized by its geometric, inertia and aerodynamic properties. Aircraft aerodynamics is usually expressed in terms of the six aerodynamic coefficients ($C_L$, $C_D$, $C_Y$, $C_I$, $C_m$, $C_n$) which are functions of aerodynamic derivatives and geometric parameters of the aircraft. The aerodynamic coefficients are computed by summing up all the effects as:

$$C_L = C_{L0} + KC_L^2$$

$$C_D = C_{D0} + KC_L^2$$

$$C_I = C_{I0} + KC_L^2$$

$$C_m = C_{m0} + KC_L^2$$

$$C_n = C_{n0} + KC_L^2$$

where $C_L$ is the lift coefficient, $C_D$ is the drag coefficient, $C_I$ is the pitching moment coefficient, $C_m$ is the rolling moment coefficient, $C_n$ is the yawing moment coefficient, $K$ is a constant, and $C_L$ is the lift coefficient.

The aerodynamic derivatives are computed by:

$$\alpha + \frac{C_{ma}}{C_m} \left( \frac{x_{cg}}{S} \right) - \frac{C_{fus}}{C_m} \alpha - \frac{C_{HT}}{C_m} \eta r - \frac{C_{mac}}{C_m} \eta$$

$$\alpha + \frac{C_{ma}}{C_m} \left( \frac{\beta}{S} \right) - \frac{C_{fus}}{C_m} \beta - \frac{C_{HT}}{C_m} \eta r - \frac{C_{mac}}{C_m} \eta$$

where $C_{ma}$ is the lift curve slope of wing and horizontal tail, $\alpha$ is the angle of attack, $\beta$ is the bank angle, $\eta$ is the elevator effectiveness, $r$ is the rudder effectiveness, $S$ is the wing area, $C_m$ is the pitching moment coefficient, $C_fus$ is the fuselage lift coefficient, $C_{HT}$ is the horizontal tail lift coefficient, $C_{mac}$ is the pitching moment coefficient of the fuselage, and $C_{m}$ is the pitching moment coefficient of the horizontal tail.

Stability and control derivatives of an aircraft can be estimated through standard empirical relations provided in many flight mechanics and design textbooks ([2], [19], [20], [21], [22], [23], [24]). These empirical relations express aerodynamic derivatives as functions of geometric parameters of an aircraft. For example, expressions for aerodynamic derivatives $C_{Lq}$, $C_{nq}$, and $C_{nq}$, taken from Ref.[21] are:

$$C_{Lq} = 2 \eta V_{HT} a_t$$

$$C_{nq} = -\eta V_{HT} \delta e a_t$$

$$C_{nq} = -\frac{S VT}{S b} \eta \delta r a_v$$

where $S VT$ denotes the vertical tail size, $a_v$ is the lift curve slope of vertical tail, $\delta e$ refers to the elevator effectiveness parameter, and $\delta r$ represents the rudder effectiveness parameter. The bifurcation theoretic design methodology uses standard empirical relations for computation of stability and control derivatives; values of the computed aerodynamic derivatives vary with changes in aircraft design.
geometry and flight conditions. In the present work, the aerodynamic derivatives vary with changes in vertical tail size and c.g position of the aircraft.

Preliminary Design Data

To demonstrate the usefulness of the bifurcation theoretic design methodology, conceptual design problem of a six-seater general aviation airplane [taken from Ref. 25] is addressed in this paper. The aircraft is a low-wing configuration with conventional empennage. Wing is designed as a trapezoidal planform with a taper ratio of 0.5; NACA 23018 airfoil section is used for the entire wing. Symmetrical NACA 0012 airfoil section is employed for the horizontal and vertical tails design. The aircraft is powered by a 360 hp Textron Lycoming piston engine driving a three bladed propeller in tractor configuration. Some of the key design parameters for the aircraft are provided in Tables-1 and 2.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Numerical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum lift coefficient $C_{L_{max}}$ with zero flaps</td>
<td>1.44</td>
</tr>
<tr>
<td>$(L/D)_{max}$</td>
<td>14</td>
</tr>
<tr>
<td>Wing area $(S)$</td>
<td>176 ft$^2$</td>
</tr>
<tr>
<td>Wing span $(b)$</td>
<td>35.27 ft</td>
</tr>
<tr>
<td>Mean aerodynamic chord $(c)$</td>
<td>5.17 ft</td>
</tr>
<tr>
<td>Location of wing aerodynamic center $(x_{air_w})$ from the aircraft nose</td>
<td>7.69 ft</td>
</tr>
<tr>
<td>Wing dihedral angle</td>
<td>5 deg</td>
</tr>
<tr>
<td>Zero-lift drag coefficient $(C_{D0})$</td>
<td>0.017</td>
</tr>
<tr>
<td>Drag-due-to-lift coefficient $(K)$</td>
<td>0.075</td>
</tr>
<tr>
<td>Wing aspect ratio $(AR)$</td>
<td>7.07</td>
</tr>
<tr>
<td>Aspect ratio for horizontal tail</td>
<td>4</td>
</tr>
<tr>
<td>Moment arm of horizontal tail $(l_{HT})$ for $X_{cg}$ at 28.5% of MAC</td>
<td>17.13 ft</td>
</tr>
<tr>
<td>Horozontal tail area $(S_{HT})$</td>
<td>37.2 ft$^2$</td>
</tr>
<tr>
<td>Aspect ratio for vertical tail</td>
<td>1.5</td>
</tr>
<tr>
<td>Moment arm of vertical tail $(l_{VT})$ for $X_{cg}$ at 28.5% of MAC</td>
<td>16 ft</td>
</tr>
<tr>
<td>Fuselage length</td>
<td>25.9 ft</td>
</tr>
<tr>
<td>Engine power, supercharged to 18000 ft</td>
<td>360 hp</td>
</tr>
<tr>
<td>Propeller efficiency</td>
<td>0.8</td>
</tr>
<tr>
<td>Propeller diameter</td>
<td>6.53 ft</td>
</tr>
<tr>
<td>Location of aircraft neural point $(x_{n})$ from the aircraft nose</td>
<td>8.387 ft</td>
</tr>
<tr>
<td>Take-off gross weight $(W)$</td>
<td>4100 lb</td>
</tr>
</tbody>
</table>

The Bifurcation Theoretic Integrated Design Methodology

A bifurcation theoretic integrated design methodology, recently formulated by the authors [26], is presented in this section. To formulate the methodology, an aircraft design framework is developed within AUTO 2000 bifurcation and continuation software. The EBA method is implemented within the framework for solving constrained bifurcation analysis problem. An important feature built within the design framework is the use of various aircraft design variables, like tail area, c.g location, etc., as system parameters. This feature is very helpful in sizing of design parameters, and in readily analyzing the effect of variation in design parameters on attainable trims, stability and performance of an aircraft.

The bifurcation theoretic integrated design methodology can be lucidly explained through the flowchart shown
in Fig.1. It can be noted from the flowchart that the design methodology involves the following steps:

1. A design parameter of interest is first selected. The selected design parameter should represent some of the geometrical features of the aircraft. For example, rudder area, vertical tail area, horizontal tail area, aspect ratio of wing, etc., can be considered as one of the candidate design parameters for the design methodology.

2. Using the standard empirical relations, stability and control derivatives for the aircraft are then computed for a fixed value of the chosen design parameter.

3. Knowing the values of stability and control derivatives from step 2, aerodynamic force coefficients ($C_L$, $C_D$, $C_Y$) and aerodynamic moment coefficients ($C_l$, $C_m$, $C_n$) for the aircraft can be obtained.

4. Using the aerodynamic derivatives and aerodynamic coefficients computed in steps 2 and 3, a relevant constrained bifurcation analysis problem representing a desirable flight condition is solved in step 4 of the methodology with the help of the EBA method. Output from constrained bifurcation analysis procedure is bifurcation diagrams showing the aircraft performance, stability and attainable trim points in the specified constrained flight condition. The procedure from steps 1-4 can be carried out in following two distinct manners:

4.a) Performance and stability requirements for the aircraft in a relevant constrained flight condition can be specified as constraints and steps 1-4 be repeated for different values of the design parameter so as to find a feasible value of the design parameter satisfying the performance and stability specifications. Thus, sizing of the chosen design parameter can be efficiently done using the design methodology.

4.b) The design parameter of interest can be kept at a fixed value and steps 1-4 be performed for different constrained flight motions, like level flight, constant speed climb or descent, etc., to evaluate the aircraft performance and stability in a range of flight scenarios. Hence, a complete analysis of the aircraft performance and stability for a designed configuration can be effectively carried out with the help of the methodology.

5. Bifurcation diagrams obtained in step 4(b) provide attainable trim points for the aircraft in different flight conditions. These bifurcation diagrams can be utilized as reference maps to efficiently select the trim points between which the aircraft trajectory should be switched so as to achieve a desired maneuver. A reference profile between the selected trim points is then fed as a command input to a controller; the controller for maneuver design is to be formulated so as to track or regulate the controlled variables for the maneuver.

Thus, the complete conceptual design cycle from configuration sizing of the aircraft to maneuver design can be investigated in a coherent manner with the help of the design methodology. To illustrate the usefulness of the design methodology, sizing of vertical tail of the six-seater general aviation aircraft is first carried out with the objective of satisfying the lateral-directional flying and handling qualities specifications. Next, performance and stability of the six-seater aircraft in turning flight condition is evaluated by computing bifurcation diagrams of the

<table>
<thead>
<tr>
<th>Table-2 : Estimated Values of Airplane Parameters</th>
<th>Numerical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero-lift angle of attack ($\alpha_{L=0}$)</td>
<td>-1.2 deg</td>
</tr>
<tr>
<td>Pitching moment coefficient about aerodynamic center $C_{mac}$</td>
<td>-0.005</td>
</tr>
<tr>
<td>Elevator effectiveness parameter ($\tau_{\delta e}$)</td>
<td>0.5</td>
</tr>
<tr>
<td>Aileron effectiveness parameter ($\tau_{\delta a}$)</td>
<td>0.4</td>
</tr>
<tr>
<td>Rudder effectiveness parameter ($\tau_{\delta r}$)</td>
<td>0.55</td>
</tr>
<tr>
<td>Horizontal tail incidence ($\iota$)</td>
<td>-1 deg</td>
</tr>
<tr>
<td>Roll inertia ($I_x$)</td>
<td>2475 slug-ft²</td>
</tr>
<tr>
<td>Pitch inertia ($I_y$)</td>
<td>3083 slug-ft²</td>
</tr>
<tr>
<td>Yaw inertia ($I_z$)</td>
<td>4529 slug-ft²</td>
</tr>
</tbody>
</table>
aircraft in a steady turn. Lastly, maneuverability assessment of the aircraft in turning flight is done by constructing a turn maneuver for the aircraft using information available from the computed bifurcation diagrams.

**Sizing of Vertical Tail**

Usefulness of the EBA technique for optimally sizing the vertical tail of an aircraft was demonstrated by the authors in [18]. A few of the important results from Ref.[18] will be reproduced in this subsection to highlight the effectiveness of the bifurcation theoretic design methodology for configuration sizing of an aircraft. The sizing problem can be stated as follows:

**Design objective:** size the vertical tail of the aircraft subject to the aircraft satisfying level 1 lateral-directional flying qualities specifications for a class I airplane in category B flight phase.

**Design variable:** Vertical tail volume ratio $TR$.

To meet the above design objective, the design methodology is utilized to carry out bifurcation analysis of the aircraft in level flight conditions for different vertical tail sizes. Raymer [2] recommends a vertical tail volume ratio of 0.04 for the class of airplane studied in this work. Various vertical tail volume ratios (0.02, 0.03, 0.04 and 0.05) around the recommended value are investigated here to find the best vertical tail size satisfying the handling qualities requirements. Effects of changes in vertical tail size on the aircraft drag and gross weight are neglected for preliminary analysis of this work. Stability parameters of different flight modes for various vertical tail sizes are then calculated using respective eigenvalues available from bifurcation analysis results. A comparison of stability parameters with the corresponding values suggested in flying qualities specifications (Table-3) is finally made to find the best vertical tail size satisfying the design requirements.

To size the vertical tail, the first step of the EBA method is used to solve the following constrained system:

$$\dot{x} = f(x, u, p)$$

$$\phi = 0, \beta = 0, \gamma = 0$$

(5)

where $x = [V, \alpha, \beta, p, q, r, \phi, \theta]^T$ is the vector of state variables, superscript ‘$T$’ indicating the transpose; $u = [\delta e]$ is the continuation parameter; and $p = [\eta_p, \delta a, \delta r, TR, X_{cg}, H]$ represents the system parameters, $H$ being the aircraft altitude. Vertical tail volume ratio $TR$ is defined here with respect to $X_{cg}$ at 28.5% of mean aerodynamic chord (MAC). In order to satisfy the three constraints ($\phi = 0, \beta = 0, \gamma = 0$) of Eqn.(5), parameters $\eta_p$, $\delta a$ and $\delta r$ are freed, keeping the other parameters in $p$ as fixed. Solution of the system (5) provides parameter schedules for the free parameters required to regulate the imposed constraints of Eqn.(5).

Using the parameter schedules obtained in step 1, the following system of equations is solved in step 2 of the EBA method.

$$\dot{x} = f(x, u, p_1(u), p_2)$$

(6)

where $p_1(u) = [\eta_p(u), \delta a(u), \delta r(u)]$ are parameter schedules for the free parameters and vector $p_2 = [TR, X_{cg}, H]$ denotes the fixed parameters.

Bifurcation analysis procedure of Eqns.(5-6) is repeated for four different $TR$ (0.02, 0.03, 0.04 and 0.05) and two distinct c.g locations (forward c.g at 15.8 % of MAC, aft c.g at 33.4 % of MAC). Stability parameters of lateral-directional modes are computed using the respective eigenvalues available from bifurcation analysis results. Tables-4 and 5 show stability parameters for different values of $TR$. Stability parameters are presented for forward c.g location only, results for aft c.g location show a similar trend and are therefore not included. It can be inferred from the results of Table-4 that dutch roll mode is unstable for $TR=0.02$. Further, it is evident from Table-5 that time to double $t_2$ of spiral mode for $TR=0.05$ is lesser than the minimum value recommended in flying qualities specifications. Hence, $TR=0.02$ and $TR=0.05$ are not the feasible solutions to vertical tail sizing problem. Again, it can be noted from the results of Table-5 that both the vertical tail volume ratios of 0.03 and 0.04 meet the flying qualities specifications of Table-3. Hence, a lower $TR$ of 0.03 represents the best vertical tail size fulfilling the flying qualities specifications.

**Table-3 : Flying Qualities Specifications [27]**

(Class I Airplane; Category B Flight Phase)

<table>
<thead>
<tr>
<th>Level</th>
<th>Dutch Roll Minimum $\zeta$</th>
<th>Dutch Roll Minimum $\omega_{\phi}$ (rad/s)</th>
<th>Spiral Mode Minimum $t_2$ (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.08</td>
<td>0.4</td>
<td>20</td>
</tr>
</tbody>
</table>

![Table-3: Flying Qualities Specifications](image-url)
Aircraft Performance and Stability Analysis

The bifurcation theoretic design methodology can be efficiently used to analyze the performance and stability of an aircraft in a variety of flight conditions. For illustration, the methodology will be applied here to study the performance and stability of the six-seater aircraft in a steady turn. Further, the effect of variations in vertical tail size and c.g location on the performance and stability of the aircraft will be examined.

It is desired to assess the turn performance of the aircraft in a zero sideslip, constant speed ($V = 190$ ft/s) turn. Hence, the following constrained system is solved using the first step of the EBA method.

$$\dot{x} = f(x, u, p) = 0$$

$$\beta = 0, V = 190 \text{ ft/s} \text{ (Mach number} = 0.17)$$ \hspace{1cm} (7)

where $x = [V, \alpha, \beta, p, q, r, \phi, \theta]^T$ is the vector of state variables, $u = [\delta e]$ is the continuation parameter, and $p = [\eta_f, \delta \alpha, \delta r, TR, X_{cg}^g, H]$ represents the system parameters.

To satisfy the two constraints of Eqn.(7), $\delta \alpha$ and $\delta r$ parameters are freed from vector $p$. Parameters $\eta_f (= 0.35)$, $TR$, $X_{cg}^g$ and $H$ remain fixed during the continuation run. Continuation procedure is carried out up to an angle of attack (AOA) of 0.175 rad only so as to restrict bifurcation analysis within the region in which empirical relations for aerodynamic derivatives hold well. Initial condition for continuation of the constrained system (7) is taken from bifurcation analysis results of level flight condition computed in Section (Sizing of Vertical Tail). The state vector and control inputs for this initial condition, satisfying constraints of Eqn.(7), are:

$$[V, \alpha, \beta, p, q, r, \phi, \theta] = [190, 0.09767, 0, 0, 0, 0, 0.09767]$$

$$\eta_f = 0.35, \delta e = -0.0687, \delta \alpha = 0, \delta r = 0$$

Next, the second step of the EBA method is carried out to analyze stability of the steady turn solutions computed in the first step. Thus, solve:

$$\dot{x} = f(x, u, p_1(u), p_2)$$ \hspace{1cm} (8)

where $p_1(u) = [\delta \alpha(u), \delta r(u)]$ are parameter schedules for the free parameters obtained from the first step of the EBA method, and vector $p_2 = [\eta_f = 0.35, TR, X_{cg}^g, H]$ denotes the fixed parameters.

Constrained bifurcation analysis procedure of Eqns.(7-8) is performed for three different values of $TR$ (0.03, 0.04, 0.05) and two different $X_{cg}^g$ (forward c.g at 15.8% of MAC, aft c.g at 33.4% of MAC) using the EBA-based design program. Parameter schedules of aileron and rudder obtained from the solution of the first step of the EBA method are shown in Figs.2-3: schedules for aileron do not vary with changes in vertical tail volume ratio. Further, from Fig.3, it can be noted that higher rudder deflections are required to trim the aircraft in a particular bank angle for larger vertical tail size: this happens because the relative increment in yaw damping parameter $C_{ philosopher}$, which generates an adverse yawing moment during a turn, is more than the corresponding increment in rudder control power $C_{ phi phi}$, for larger vertical tail size. Bifurcation diagrams of steady turn solutions for the three values of $TR$ are presented in Figs. 4-6. Fig.4 shows bifurcation diagram of steady turn solutions for $TR=0.03$. All the trim points are stable for this case for both the forward and aft locations of c.g of the aircraft. Bifurcation diagram of steady turn solutions for $TR=0.04$ are presented in Fig.5. The entire solution branch is seen to be stable for this case.

### Table-4 : Eigenvalues for Dutch Roll Mode at ISA Sea Level; TR = 0.02 [18]

<table>
<thead>
<tr>
<th>AOA (rad)</th>
<th>c.g. (% MAC)</th>
<th>Altitude (ft)</th>
<th>Real Part of Root</th>
<th>Imaginary Part of Root</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.035</td>
<td>15.8</td>
<td>Sea level</td>
<td>0.3522</td>
<td>± 0.0977</td>
</tr>
<tr>
<td>0.175</td>
<td>15.8</td>
<td>Sea level</td>
<td>0.0177</td>
<td>± 0.5579</td>
</tr>
</tbody>
</table>

### Table-5 : Stability Parameters at ISA Sea Level; c.g = 15.8% MAC [18]

<table>
<thead>
<tr>
<th>TR</th>
<th>AOA (rad)</th>
<th>Dutch Roll $\omega_0$ (rad/s)</th>
<th>Dutch Roll $\zeta$</th>
<th>Spiral Mode $t_{1/2}$ (s)</th>
<th>Spiral Mode $t_2$ (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.03</td>
<td>0.035</td>
<td>1.512</td>
<td>0.155</td>
<td>11.21</td>
<td>-</td>
</tr>
<tr>
<td>0.03</td>
<td>0.175</td>
<td>1.054</td>
<td>0.213</td>
<td>26.02</td>
<td>-</td>
</tr>
<tr>
<td>0.04</td>
<td>0.035</td>
<td>2.325</td>
<td>0.153</td>
<td>25.84</td>
<td>-</td>
</tr>
<tr>
<td>0.04</td>
<td>0.175</td>
<td>1.426</td>
<td>0.219</td>
<td>-</td>
<td>19.43</td>
</tr>
<tr>
<td>0.05</td>
<td>0.035</td>
<td>2.952</td>
<td>0.16</td>
<td>40.3</td>
<td>-</td>
</tr>
<tr>
<td>0.05</td>
<td>0.175</td>
<td>1.735</td>
<td>0.222</td>
<td>-</td>
<td>11.56</td>
</tr>
</tbody>
</table>
as well. Turn solutions for $TR = 0.05$ are depicted in Fig.6. The solutions change from unstable for small values of bank angle, to stable for medium bank angles, large bank angle turn solutions are unstable in nature. Though the limits of stable and unstable solutions for $TR = 0.05$ differ for the forward and aft c.g locations, the mechanism of instability remains same for the two c.g positions: low bank angle turn solutions become unstable at a transcritical bifurcation point, whereas medium bank angle turn solutions lose stability at a Hopf bifurcation point. Fig.7 shows the corresponding AOA at which transcritical and Hopf bifurcation occurs.

Design of Aircraft Turn Maneuver

It has now been demonstrated that the bifurcation theoretic design methodology can aid in configuration sizing and also prove useful for concurrent performance and stability analysis of an aircraft. Bifurcation diagram computed as part of the performance and stability evaluation studies of an aircraft provides a map containing all the trim states of the aircraft in a steady constrained flight condition. Using information from bifurcation diagrams of the aircraft in different steady constrained flight conditions, one can efficiently analyze maneuverability of the aircraft by constructing various kinds of maneuvers possible by the aircraft. Fig.8 pictorially explains the idea of maneuver design. Bifurcation theoretic design code provides trim points for the maneuver; these trim points define the reference profile for the maneuver. Next, the reference maneuver profile is fed to a closed-loop simulation model of the aircraft. Matlab Simulink software is used to develop the simulation model in this work. The controller within the closed-loop model then generates the necessary control input commands to simulate the maneuver. A facility for visualizing the maneuver is also built within the design program using open source Flight Gear program.

To exemplify the usefulness of the bifurcation theoretic design methodology in constructing maneuvers of an aircraft, design of a constant speed turn maneuver for the six-seater aircraft with $TR=0.03$ is pursued here. Objective of turn maneuver to be designed is to transition the aircraft from a straight and level symmetric flight condition to a turning flight condition corresponding to trim point (B) of Fig.4, while holding a constant speed ($V=190$ ft/s) and restricting sideslip to zero. Because constant speed and zero sideslip have to be maintained during turn maneuver, level flight trim point (A) of Fig.4 is chosen as initial operating point for the maneuver.

The state vector and control inputs for initial trim point (A), also used as starting point for constrained bifurcation analysis of the augmented system (7), are:

$$\begin{align*}
V, \alpha, \beta, p, q, r, \phi, \theta &= [190, 0.09767, 0, 0, 0, 0, 0.09767] \\
\eta_f &= 0.35, \delta e = -0.0687, \delta a = 0, \delta r = 0
\end{align*}$$

And the state vector and control inputs for final trim point (B), as obtained from constrained bifurcation analysis of steady turn in Section (Aircraft Performance and Stability analysis) are:

$$\begin{align*}
V, \alpha, \beta, p, q, r, \phi, \theta &= [190, 0.155, 0, -0.0104, 0.137, 0.125, 0.83, 0.056] \\
\eta_f &= 0.35, \delta e = -0.161, \delta a = 0.012, \delta r = -0.007
\end{align*}$$

Knowing the initial and final trim points for turn maneuver, the maneuver is designed so as to switch the aircraft state from initial trim point (A) to final trim point (B). A nonlinear sliding mode controller is used to transition the aircraft between the trim points. The closed-loop system with sliding mode controller is shown in Fig.9. Starting from left on Fig.9, a desired maneuver profile is fed as reference input to the closed-loop system. Next, the reference input is compared with output vector $y$ to calculate error vector and its integral. Using the calculated error vector, integral of error vector, and a feedback signal of state variables, the sliding mode controller block computes control input command. The computed control input from the controller is passed through control surfaces’ saturation and rate constraint blocks to provide command input to the aircraft model. A simple anti-windup logic is also incorporated within the closed-loop system to prevent integral windup.

Closed-loop simulation results for constant speed, zero sideslip turn maneuver of the six-seater aircraft are displayed in Fig.10. A full 6-DOF, nonlinear mathematical model of the aircraft is used for simulating the maneuver. ISA sea level conditions are assumed for ambient atmospheric variables, and the effect of density variation due to change in altitude of the aircraft during the maneuver is neglected for simulation. Position limits of different control effectors are provided in Table-A (see Appendix). Effect of control surface rate saturation is ignored for simulating the maneuver; control surfaces’ rate saturation limits can however affect the stability of closed-loop system. To start the maneuver, the aircraft is first considered to be flying in a straight and level symmetric flight condition corresponding to trim point (A) of Fig.4 for initial 10
seconds. Next, the sliding mode controller is activated at time=10 seconds to switch the aircraft to trim point (B) of Fig.4 in a step profile. Errors in the three output variables ($y = [\alpha, \beta, \mu]^T$) reduce to zero within a duration of 10 seconds from the start of control action, as is evident from the error plots of Fig.10d. Body axis rates, shown in Fig.10b, approximately stabilize to their respective steady state values within 10 seconds. From Fig.10a, fluctuations in Mach number and sideslip are found to be minimal. Thrust parameter $\eta_T$ is kept at a constant value of 0.35 during turn maneuver, as is shown in Fig.10e. Time histories of control inputs from the sliding mode controller are provided in Figs.10e-10f. A nose-up elevator command is given to increase AOA to required final steady state value. An initial right aileron command input is provided to roll the aircraft to right, and a right rudder input is simultaneously applied to restrict sideslip to zero during the turn. Radius of turn for the aircraft can be noted to be around 1000 ft from Fig.10g. Variation in the aircraft heading $\psi$ is shown in Fig.10h, from which time taken to complete the longitudinal loop can be found to be around 33.3 seconds; time to turn closely matches with the time obtained using turn rate ($\psi = 0.186 \text{ rad}$) computed from body axis rate components of trim point (B).

In the maneuver simulation example of Fig.10, aerodynamic derivatives for the controller vary with AOA of the aircraft; this considerably complicates the expression for sliding mode controller. As sliding mode control is generally known to be robust to parametric uncertainties, we therefore attempt to simulate the turn maneuver by using constant values of aerodynamic derivatives. This will also illustrate the robustness of the sliding mode controller. To simulate the maneuver, stability and control derivatives for the controller are fixed at their respective values for flight condition (A). All the constants used in the sliding mode algorithm remain same as that used for the maneuver simulation results of Fig.10. Numerical simulation results using constant values of stability and control derivatives are provided in Fig.11. From Fig.11a, amplitude of oscillations in AOA can be noticed to be larger as compared to the corresponding results of Fig.10, however, fluctuations in sideslip remain negligibly small. From Fig.11e, larger control activity is observed in elevator control channel for this uncertain case. Higher control activity seen in longitudinal channel arises because of lesser downwash angle ($\varepsilon$) assumed in the expression of the controller, as trim point (A) is at lower angle of attack than the final trim point (B). As a result of lower downwash angle taken in the expression of sliding mode controller, the effective nose-down moment available from horizontal tail is assumed to be higher by the controller resulting in more severe control activity in elevator and hence larger settling time for AOA. Larger oscillations in AOA result in higher variations in pitch rate ($\dot{\psi}$), as visible from Fig.11b. Time variation of remaining aircraft variables shown in Fig.11 is identical to that noted in corresponding plots of Fig.10. Fig.12 shows a screenshot of the turn maneuver as viewed in Flight Gear using a representative aircraft.

Conclusions

A bifurcation theoretic integrated design methodology has been presented in this paper. It has been demonstrated that the methodology can significantly aid in various stages of conceptual design. A configuration sizing example of tail design is first illustrated to highlight the usefulness of the technique. Next, the efficacy of the methodology for detailed performance and stability analysis is successfully illustrated. Lastly, the effectiveness of the methodology for maneuverability assessment of the aircraft is demonstrated by constructing an aircraft maneuver.

Some of the salient features of the design methodology can be summarized as below:

- The methodology gives a design framework in which configuration sizing, trim and stability analysis, performance evaluation, and maneuverability assessment can all be studied in a coherent fashion. Configuration sizing is achieved by solving an inverse constrained bifurcation analysis problem, i.e., given a set of constraints on the performance and/or stability requirements of an aircraft, size the aircraft so as to satisfy the stipulated constraints. Maneuverability of the aircraft is examined via maneuver design with the help of a controller.

- The methodology can be extremely useful for efficient design of aircraft and unmanned air vehicles with desired handling qualities, performance and maneuver characteristics.

- The methodology can serve as an important design tool for control prototyping.

- Being based on empirical relations, the methodology can be employed as a pedagogical tool to enliven the subjects related to aircraft design and control.
The proposed design methodology makes use of empirical relations for estimation of various aerodynamic stability and control derivatives. And, because these empirical relations for aerodynamic derivatives are valid within low angle of attack region only, the methodology in its present form is mainly applicable for design studies in low angle of attack regime. Future work in this area should involve the following:

- The design methodology should be suitably extended to high angle of attack flight regime.
- Currently, there is a great deal of interest in developing reconfigurable control algorithms that allow safe flight operation of an aircraft in the event of structure or system failure. In this direction, expansion of the methodology for design of robust aircraft that can safely handle structural or system damage without any need for control reconfiguration may be explored.

References


Appendix

Rigid Body Aircraft Flight Dynamics Equations

The rigid body aircraft flight dynamics equation [29].

\[ \dot{V} = \frac{1}{m} \left[ T \eta \cos \alpha \cos \beta - \frac{1}{2} \rho V^2 S C_D (\alpha, q, \delta_e) - mg \sin \gamma \right] \]

\[ \dot{\alpha} = q - \frac{1}{\cos \beta} \left[ (p \cos \alpha + r \sin \alpha) \sin \beta + \frac{1}{m} V \right] \]

\[ \dot{\beta} = \frac{1}{mV} \left[ -T \eta \cos \alpha \sin \beta + \frac{1}{2} \rho V^2 S C_L (\alpha, \beta, p, r, \delta_a, \delta_r) \right] \]

\[ \dot{\phi} = \frac{I_y - I_z}{I_x} \frac{q}{r} + \rho \frac{V^2}{2 I_y} S \bar{C}_m (\alpha, q, \delta_e) \]

\[ \dot{q} = \frac{I_z - I_x}{I_y} \frac{p}{r} + \rho \frac{V^2}{2 I_y} S \bar{C}_m (\alpha, \beta, p, r, \delta_a, \delta_r) \]

\[ \dot{\gamma} = \frac{I_x - I_y}{I_z} \frac{p}{r} + \rho \frac{V^2}{2 I_z} S \bar{C}_n (\alpha, \beta, p, r, \delta_a, \delta_r) \]

\[ \dot{\psi} = \frac{q}{\cos \theta} \cos \phi \tan \theta + r \cos \phi \tan \theta \]

\[ \dot{\theta} = q \cos \phi - r \sin \phi \]

\[ \dot{\psi} = \frac{q}{\cos \theta} + r \cos \phi \]
Wind axis Euler angles \( (\mu, \gamma) \) and body axis Euler angles \( (\phi, \theta) \) are related by the following kinematic relations:

\[
\sin \gamma = \cos \alpha \cos \beta \sin \theta - \sin \beta \sin \phi \cos \theta - \sin \alpha \cos \beta \cos \phi \cos \theta
\]

\[
\cos \mu \cos \gamma = \cos \alpha \sin \beta \sin \theta + \cos \beta \sin \phi \cos \theta - \sin \alpha \sin \beta \cos \phi \cos \theta
\]

\[
\sin \mu \cos \gamma = \sin \alpha \sin \theta + \cos \alpha \cos \phi \cos \theta
\]

\[
\cos \mu \cos \gamma = \sin \alpha \sin \theta + \cos \alpha \cos \phi \cos \theta
\]

<table>
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<tr>
<td>Elevator, ( \delta e )</td>
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<tr>
<td>Aileron, ( \delta a )</td>
</tr>
<tr>
<td>Rudder, ( \delta r )</td>
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</tbody>
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Fig. 2 Aileron Schedules for Steady Turn with Different TR at ISA Sea Level; Mach Number = 0.17

Fig. 3 Rudder Schedules for Steady Turn with Different TR at ISA Sea Level; Mach Number = 0.17

Fig. 4 Bifurcation Diagram of \( \phi \) with \( \delta e \) for Steady Turn with TR = 0.03, Stable Point
Fig. 5 Bifurcation Diagram of $\phi$ with $\delta e$ for Steady Turn with $TR = 0.04$; , Stable Point

Fig. 6 Bifurcation Diagram of $\phi$ with $\delta e$ for Steady Turn with $TR = 0.05$; , Stable Point, ---- , Unstable Point; , Transcritical Bifurcation, , Hopf Bifurcation

Fig. 7 Bifurcation Diagram of Angle of Attack with $\delta e$ for Steady Turn with $TR = 0.05$; , Stable Point, ---- , Unstable Point; , Transcritical Bifurcation, , Hopf Bifurcation

Fig. 8 Diagram Illustrating Maneuver Design Procedure

Fig. 9 Simulink Block Diagram for the Closed-Loop System with Sliding Mode Controller [28]
Fig. 10 Time History of Aircraft State Variables and Control Inputs for Constant Speed Turn Maneuver of the Six-Seater Airplane Using the Sliding Mode Controller; TR = 0.03; Forward c.g.
Fig. 11 Time History of Aircraft State Variables and Control Inputs for Constant Speed Turn Maneuver Using the Sliding Mode Controller with Aerodynamic Derivatives Fixed at Flight Condition (A) of Fig. 4; TR = 0.03; Forward c.g
Fig. 12 Screenshot of the Aircraft in Turn Maneuver