INTERLAMINAR STRESS CALCULATION IN COMPOSITE LAMINATES USING AN 8-NODED BRICK ELEMENT BASED ON MIXED FINITE ELEMENT FORMULATION

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Abstract

An eight-noded brick element is developed for accurate estimation of interlaminar stresses in composite laminates. This element incorporates the three transverse stresses $\sigma_z$, $\sigma_{yz}$ and $\sigma_{xz}$ as the degree of freedoms apart from the three displacements ($u$, $v$ and $w$). The displacements are calculated as usual from the principle of virtual work, and the stresses are interpolated independently in the constitutive equation. In composite laminates the shear and normal stresses are continuous across the laminates. This stress continuity can be imposed using continuous shape functions for stress components. Studies are done for various laminates and results are compared with available solutions from the literature.

Keywords: Laminated composites, Mixed finite element formulation, Interlaminar stresses, 8-noded element.

Introduction

Composite materials are used extensively in aerospace and ground applications due to high strength to weight and stiffness to weight ratio. Delamination is one of the major modes of failure in composites. Due to their characteristics of low transverse shear stiffness, composite laminates exhibit significant transverse shear deformation for smaller span to depth ratio. The available classical plate theory (CPT) and first order shear deformation theory (FSDT) can predict the global response such as gross deflection, natural frequency of the structure well. This is especially true for thin laminates.

As the laminate thickness to span ratio increases these theories are inadequate in the prediction of transverse stress accurately. For critical components the assessment of stress in localized regions is necessary where delamination can take place.

Accurate estimation of interlaminar stresses in composites is subject of investigation by many researchers. Methods like elasticity solution, CPT, FSDT, and higher order theory based finite elements are used to solve the problem involving composite construction. Pagano [1] developed exact elasticity solution for rectangular laminates under sinusoidal loading. The results were compared with CPT solution and CPT was found to be inaccurate at low span to dept ratio. Pagano et. al [2], have also used finite differences for estimating interlaminar stresses under in-plane loading. Reddy [3] developed a third order theory which allowed quadratic variation of transverse shear strains and vanishing of transverse shear at top and bottom. The accuracy in stress calculation was better at increased computational effort. Reddy et. al [4] developed analytical solutions for laminates based on his generalized plate theory. The results are obtained for thick laminates and compared with Pagano’s [1,5] elasticity solutions. Finite elements technique is a powerful tool for solving complex geometry and can incorporate any of the theory. Displacement based finite element was used to determine shear stresses in laminates by Wei and Zhao [6] under inplane loading. Liu and Jou [7] extended the displacement-based element by introducing transverse stress variables as degree of freedom. The constitutive relations for the transverse stress are introduced as constraint to the virtual work. The element is used for two-dimensional problems and was used in-plane loading. Wu and Yen [8] developed a mixed finite element based on local higher
order lamination theory where they proposed a layer wise cubic approximation for in-plane displacement. Also three rotational and five higher order functions were used as generalized degrees of freedom. Marimuthu et. al [9] used the mixed formulation to develop a 20-noded brick element to estimate interlaminar stresses under transverse loading. Finite element based on partial hybrid stress method was used by Yong and Cho [10] for interlaminar stress calculation.

The continuity of transverse stress is an important condition to be satisfied at the interlaminar boundary. In the equivalent single layer plate theories such as classical laminate plate theory (CPT) and First order shear deformation theory (FSDT) assumption is made for displacements to be continuous function of thickness. This leads to transverse strains being continuous and since the material modulus in general is not equal between adjacent layers, the transverse stress becomes discontinuous. Also for displacement based finite element models the stress being the derived quantity is generally discontinuous especially for lower order elements. Hence, presently a technique is adopted in which transverse stresses (σx, σyz and σxz) are interpolated separately using Lagrange shape functions and are introduced as a nodal variable. Continuous shape functions are used for the interpolation of stress variables to impose the condition of continuous transverse stress across the layers. Also traction free surface condition can be imposed on the top and bottom surface of laminate as transverse stress is nodal variable.

Formulation

In the present formulation the displacements are derived from virtual work equation in the usual way i.e.,

\[ \int_v \sigma \delta \varepsilon^T dV - \int_v b \delta u^T dV - \int_s t \delta u^T dS = 0 \]

(1)

Here in the virtual work it is assumed that displacements are derived in the usual way by substituting \( \sigma = D \varepsilon \) and neglecting the body forces such Eq. (1) reduces to

\[ \int_v \varepsilon^T D \varepsilon^T dV - \int_s t \delta u^T dS = 0 \]

(2)

Transverse stresses are introduced as nodal variables by multiplying the variation in transverse stress, by difference in strains derived from displacements and constitutive equation i.e.,

\[ \int_v \delta \sigma_{TR} (\varepsilon_{TR} - S \sigma) dV = 0 \]

(3)

where \( \sigma_{TR} \) represents the three transverse stresses \( \sigma_x, \sigma_{yz} \) and \( \sigma_{xz} \) and similarly for strains. Expanding the above equations by breaking the stress into inplane and transverse Eq. (3) can be written as

\[ \int_v \delta \sigma_{TR} (\varepsilon_{TR} - S_{IN} \sigma_{IN} - S_{TR} \sigma_{TR}) dV = 0 \]

(4)

In the above equation the in-plane stresses \( \sigma_{IN} \) i.e. \( \sigma_x, \sigma_y, \sigma_{xy} \) are assumed to be derived from displacements such that \( \sigma_{IN} = D_{IN} \varepsilon \).

The constitutive relation for the three dimensional orthotropic material is given as

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\varepsilon_{xy} \\
\varepsilon_{yz} \\
\varepsilon_{xz}
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\
S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & S_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & S_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & S_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\sigma_{xy} \\
\sigma_{yz} \\
\sigma_{xz}
\end{bmatrix}
\]

(5)

First the displacements are interpolated in Eq. (2) such that the strains are obtained from the relationship \( \varepsilon = B d^i \) where B is the derivative of shape function matrix for displacements and d is the column vector for nodal displacements. Also the transverse stresses are interpolated with shape functions such that \( \sigma_{TR} = N_{\sigma} \sigma_{TR} \) where \( N_{\sigma} \)

d and \( \varepsilon \) and \( \varepsilon_{TR} \) matrix are given as

\[
N_{\sigma} = \begin{bmatrix}
N_i & 0 & 0 \\
0 & N_i & 0 \\
0 & 0 & N_i
\end{bmatrix}
\]

(6)
\[
\varepsilon = \begin{bmatrix}
\partial N_i / \partial x & 0 & 0 \\
0 & \partial N_i / \partial y & 0 \\
0 & 0 & \partial N_i / \partial z \\
\partial N_i / \partial y & \partial N_i / \partial x & 0 \\
\partial N_i / \partial z & 0 & \partial N_i / \partial y \\
\partial N_i / \partial z & 0 & \partial N_i / \partial x \\
\end{bmatrix}
\begin{bmatrix}
i \\
u \\
v \\
w \\
i \\
v \\
\end{bmatrix} \\
\]

(7)

for eight noded brick \(i = 1 \ldots 8\)

Here stress shape function \(N_\sigma\) is same as shape function for displacement \(N_u\)

After substituting the values for strain and stress in terms of nodal variables in Eqs. (2) and (4) and on simplification we get

\[
\begin{bmatrix}
\varepsilon_{x} \\
\varepsilon_{y} \\
\varepsilon_{z} \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 & \partial N_i / \partial z \\
0 & \partial N_i / \partial z & \partial N_i / \partial y \\
\partial N_i / \partial z & 0 & \partial N_i / \partial x \\
\end{bmatrix}
\begin{bmatrix}
i \\
u \\
v \\
w \\
i \\
v \\
\end{bmatrix} \\
\]

(8)

where \(i = 1 \ldots 8\) and \(\xi, \eta\) and \(\zeta\) are natural coordinates.

The sketch of the isoparametric element used in present study is shown in Fig.1.

**Numerical Examples**

Several problems for which solutions are available in literature were solved to demonstrate the accuracy of present element in determining displacements and transverse stresses. These problems involve cross ply laminates of different orientations.

**Convergence Study**

A 4-layerd beam is taken with \((0/90/90/0)\) stacking sequence to understand the displacement convergence with mesh refinement. The beam dimensions in cm are as shown in Fig.2.

The material properties for the beam are as follows:

\[E_1 = 138 \text{ GPa}, \ E_2 = E_3 = 14.5 \text{ GPa}, \ G_{12} = G_{23} = G_{31} = 5.86 \text{ GPa}, \ \nu_{12} = \nu_{13} = \nu_{23} = 0.21\]

and each layer is assumed to be of equal thickness.

The elements are varied uniformly along the length from 2 to 18 and the transverse displacement estimated. The same problem is solved in MSC NASTRAN [11]

![Fig.1 Isoparametric eight-noded element](image)

![Fig.2 Cantilever beam as an example](image)
with the eight node CHEXA element and the comparison between the two solutions is as shown in Fig.3. From the figure it is clear that both the solutions match closely and the present element possesses fast convergence.

**Bending Under Uniformly Distributed Load (UDL)**

*Example 1. Bending of Nine Layer Laminate:* A nine layered square laminate 25.4 cm wide with orientation 0/90/0/90(0)_{sym} is chosen for study. The pressure applied is 68.95 kPa and the material chosen for the laminate are

\[ \begin{align*}
E_1 &= 275 \text{ GPa}, \quad E_2 = E_3 = 6.895 \text{ GPa}, \quad G_{23} = 3.4475 \text{ GPa}, \\
G_{12} = G_{13} &= 4.137 \text{ GPa}; \quad \nu_{12} = \nu_{13} = \nu_{23} = 0.25
\end{align*} \]

The plate is simply supported at all the four edges. Here symmetry of the plate is utilized and quarter plate is modeled. A finite element mesh of 3 x 3 is chosen along the span with 9 elements along the thickness. Each layer is modeled separately (If \( h \) represents the total thickness then all the five 0° layers are thickness 0.1h and 90° layers of thickness 0.125h). Reduced integration (one gauss point) is used for sampling of transverse shear strains \( \varepsilon_{yz} \) and \( \varepsilon_{xz} \). This is done as full integration leads to stiff behavior especially at high span to depth ratio. Same problem was also solved in NASTRAN with eight node CHEXA element and modeling the layers with solid elements and using reduced integration. The present results are compared with that of NASTRAN and 20-noded brick element of Marimuthu et al [9]. The center deflections are non-dimensionalised as follows:

\[ w_c = w_0 \left( \frac{a}{2}, \frac{b}{2} \right) \left( E_2 \frac{h^3}{a^4} q_0 \right) \times 100 \quad (11) \]

The results have been taken for span to depth (a/h) ratios of 4, 10 and 100 respectively.

From the Table-1, it is clear that the present solution compares well with NASTRAN as well as reference solution. The match between the present and NASTRAN results is close. For (a/h) of 4 for the present element there is overprediction of deflection by 1.7% compared to NASTRAN. For (a/h) =10 the present element and NASTRAN overpredict the deflection by 2.8% and 2% respectively compared with reference. For (a/h) =100 both present element and NASTRAN underpredicts the deflection by about 1.1% compared to reference.

**Cylindrical Bending**

Laminated plates with simply supported edges and under cylindrical bending are investigated as next example problem to illustrate the effectiveness of present formulation. The applied loading is sinusoidal in nature given by

\[ q = q_0 \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) \]

The material properties used for all the cases in this problem are the following:

\[ \frac{E_1}{E_2} = 25, \quad E_2 = E_3 = 6.895 \text{ GPa}, \quad G_{23} = 1.379 \text{ GPa}; \quad G_{12} = G_{13} = 3.4475 \text{ GPa}; \quad \nu_{12} = \nu_{13} = \nu_{23} = 0.25. \]

Both the deflections and stresses are non-dimensional. The deflections are non-dimensionalized as in (11) and stresses as

\[ \sigma_{yz} = (h/aq_0) \sigma_{yz} \quad \text{and} \quad \sigma_{xz} = (h/aq_0)\sigma_{xz} \quad (12) \]

**Table-1 : Deflection of (0/90/0/90/0)_{sym} laminate**

<table>
<thead>
<tr>
<th>a/h</th>
<th>Present</th>
<th>NASTRAN</th>
<th>Ref [9]</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.6212</td>
<td>1.5932</td>
<td>NA</td>
</tr>
<tr>
<td>10</td>
<td>0.6131</td>
<td>0.6086</td>
<td>0.5960</td>
</tr>
<tr>
<td>100</td>
<td>0.4420</td>
<td>0.4422</td>
<td>0.4470</td>
</tr>
</tbody>
</table>

NA - Not Available
Different lay-up sequences are considered for cylindrical bending, they are

- A (0/90) laminate with different span to depth \((a/h)\) ratios.
- A (0/90/0) laminate with equal thickness layers with \((a/b) = 1\) (square).
- \((0/90/0)\) laminate with \(b=3a\).
- A (0/90/90/0) laminate with equal thickness layers and \((a/b) = 1\).

Figure 4 shows the configuration of laminate and support conditions on the edges. Simply supported (SS) conditions are as follows:

Edges : \((0, y, \pm h/2)\) and \((a, y, \pm h/2)\) \(v = w = 0\)
Edges : \((x, 0, \pm h/2)\) and \((x, b, \pm h/2)\) \(u = w = 0\)
On the top and bottom surface ; \((x, y, \pm h/2)\) \(\sigma_{yz} = 0\) and \(\sigma_{xz} = 0\)

Since the top and bottom surfaces are traction free this condition is imposed, as transverse stresses are taken as the nodal variables. For all these problems, both the transverse shear stresses in the plots are measured at edge locations i.e \(\sigma_{yz}\) stress component is estimated at \((a/2, 0, z)\) and \(\sigma_{xz}\) stress component is estimated at \((0, b/2, z)\). In all these problems, reduced integration for transverse strains \(\varepsilon_{yz}\) and \(\varepsilon_{xz}\) is done using one point gaussian integration. Also in all the cases the loading is applied in both top and bottom surface of laminate.

Example 2. Bending of \((0/90)\) square laminate: A cross-ply square laminate with two layers is considered under simply supported conditions. In the finite element idealization \(4 \times 4\) mesh is taken along span and 12 elements along the thickness to capture the stress distribution accurately.

Studies are performed for various span-to-depth ratios with sinusoidal loading. Maximum deflections are calculated and compared with the Reddy’s third order shear deformation theory (TSDT) solutions [12] and First order shear deformation theory (FSDT) solutions [12]. The results are presented in Table-2. The deflection is non dimensional as in (11). The \((a/h)\) ratios cover laminates from thick to thin geometry.

The TSDT is more accurate compared to FSDT especially for thick laminates and the present solution compares well with TSDT for thick laminate case. For example for \((a/h) = 4\) the FSDT overpredicts deflections by 7% compared to TSDT, where as the present solution overpredicts by 0.8%. For thinner laminates \((a/h) = 10, 20, 50\) the present solution underpredicts in the range of 7-10% and FSDT overpredict in the range of 1-2% compared to TSDT. So the present element predicts deflection well for thick laminates whereas for thin laminates it underpredicts the deflection.

The stress distribution obtained from the present element for interlaminar stress \(\sigma_{xz}\) is shown in Fig.5 for the span to depth ratio \((a/h)\) of 4. This span to depth ratio was chosen to display the stress predictive capability of present element for thick laminates. The stress distribution obtained from present element compares well with exact elasticity solution of Pagano [1]. The present element seems to predict slightly higher maximum stress than exact solution as manifested in Fig.5.

Example 3. Bending of \((0/90/0)\) square laminate: A three layer cross ply square laminate \((a/b=1)\) is considered with each layer of equal thickness. The finite element description for the present and subsequent case is same as that for previous \((0/90)\) laminate. The interlaminar shear stress distributions are shown in Figs. 6 and 7 for span to depth ratio \((a/h) = 4\) and compared with available elasticity solutions in literature [1] as well as the Classical Plate Theory (CPT) solutions. It is seen from the figures that the

| Table-2 : Deflection results for \((0/90)\) laminate |
|-----------------|-----------|---------|---------|
| Max deflection, \(w_{\text{max}}\) | Present | TSDT    | FSDT    |
| \(a/h\)        |          |         |         |
| 4              | 2.0154   | 1.9985  | 2.1492  |
| 10             | 1.253    | 1.216   | 1.2373  |
| 20             | 0.9972   | 1.1018  | 1.1070  |
| 50             | 0.9612   | 1.0697  | 1.0705  |
present element stress solutions match well with the Elasticity solutions whereas CPT results do not match with the exact solutions (it under-predicts the $\sigma_{yz}$ value whereas overpredicts the $\sigma_{xz}$ value). Also, the present element seems to slightly over-predict the stresses compared to exact elasticity solution by Pagano [1] particularly for $\sigma_{yz}$. This same behavior was observed previously for the (0/90) laminate. Thus for thick laminates the element is capable of predicting transverse stress which equivalent single layer theory like CPT are unable to predict.

Example 4. Bending of (0/90/0) laminate (aspect ratio $b/a=3$): Here a rectangular laminate is taken with span ratio of 3 and subjected to cylindrical bending. The stress $\sigma_{yz}$ is calculated across the thickness and compared with exact Pagano’s [1] solution. The span to depth ratio is taken to be 4 as thick laminate is of primary interest. The interlaminar stress distribution is shown in Fig. 8.
It is clear from the plot that the transverse stress estimated by present element matches well with exact elasticity solution. Like previous case the present element predicts slightly higher stress.

Example 5. Bending of (0/90/90/0) square laminate: A square laminate of four equal thickness layers is taken in this study. Both deflection and transverse stress calculations are done for this laminate and the stress distributions plotted for (a/h) = 4. The shear stress distributions are compared with available corresponding FSDT [12] stress plots. The maximum shear stress value is also compared with available exact maximum stress values [13]. The deflections are presented in Table-3.

From Table-3 it is seen that the present element deflections compare well with the exact elasticity solutions (ELS). For (a/h) = 4 the present solution overpredicts the deflections by 0.2% compared to ELS whereas TSDT underpredicts the deflection by 3%. For (a/h) = 10 the present solution underpredicts the deflection by 5% compared to ELS whereas TSDT underpredicts the deflection by 3.7%. The FSDT solution is in error with exact solution and it grossly underpredicts the deflections (12.4% and 10.8% for a/h of 4 and 10).

The non-dimensional stress values are given in Table-4 for the given solutions.

It is seen that the FSDT solution seems to overpredict the stress $\sigma_{xz}$ and underpredict $\sigma_{yz}$ compared to exact solution (ELS), whereas the present element is close to exact solution. The stress plots are shown in Fig. 9 and Fig. 10.

Both the stresses in Fig. 9 and 10 are for a/h=4 to show the capability of present element for thick laminates as in previous examples. From Table-4, it is clear that the present solution closely predicts the exact solution (ELS) compared to FSDT. The deviation in $\sigma_{xz}$ value from exact solution is more for FSDT where it over predicts the

### Table-3 : Deflection values for (0/90/90/0) laminate

<table>
<thead>
<tr>
<th>(a/h)</th>
<th>Present</th>
<th>ELS</th>
<th>TSDT [12]</th>
<th>FSDT [12]</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.958</td>
<td>1.954</td>
<td>1.894</td>
<td>1.710</td>
</tr>
<tr>
<td>10</td>
<td>0.705</td>
<td>0.743</td>
<td>0.715</td>
<td>0.663</td>
</tr>
</tbody>
</table>

+ Exact elasticity solution [13]

### Table-4 : Stress predicted for (0/90/90/0) laminate

<table>
<thead>
<tr>
<th>(q/2, 0, 0); (a/h) = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
</tr>
<tr>
<td>$\sigma_{yz}$</td>
</tr>
</tbody>
</table>

Fig. 9 Stress $\sigma_{yz}$ for (0/90/90/0) laminate

Fig. 10 Stress $\sigma_{xz}$ for (0/90/90/0) laminate
transverse stress by 18.6% compared to ELS (exact solution). Present element overpredicts the exact solution for $\sigma_{yz}$ by 4.6% whereas TSDT under-predicts by 2%. Also for stress $\sigma_{xz}$ the present solution under-predicts stress by 1.4% whereas TSDT overpredicts by 5.2%. So both present solution as well as TSDT are close to exact solution.

**Conclusions**

An eight node hexahedron is developed with three interlaminar transverse stresses as the degree of freedom apart from displacements. The element basically calculates the stresses from the constitutive equation by defining stress as independent variables. The element is numerically tested for prediction of displacement as well as interlaminar stress in composite laminates. The solutions predicted by the element seem to compare well with the available solutions in literature. The element seems to yield fairly accurate stress estimations for thick laminates which single layer theories like CPT and FSDT are unable to capture accurately.

**References**