MULTIDISCIPLINARY ANALYSIS AND DESIGN TOOLS FOR UNCERTAINTY MODELING

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Abstract

Design optimization methods have found increased application in problems of aerospace design. The interdisciplinary nature of this design problem has given rise to the field of multidisciplinary design optimization that focuses on the integration of numerically efficient analysis methods with reliable and robust design optimization algorithms. The focus of research has increasingly shifted towards the study of modeling and handling of uncertainty in different facets of the design problem. Uncertainties add to both the computational complexity and to numerical costs in multidisciplinary design. The paper discusses recent developments in computational tools for better managing the inclusion of imprecise information, including methods through which to quantify uncertainty, new formulations of the design problem that inherently contribute to lower computational costs, and numerically efficient algorithms for computing reliability measures in multidisciplinary design. The coupled nature of analysis in the design process is examined from the perspective of decomposition-based design, and appropriate problem formulations for these cases are presented.

Introduction

With the pervasive availability of high-speed digital computers, simulation-based design has assumed new significance in the process of design and development of new aerospace systems. Physics and math based simulations for structural analysis, fluid dynamics, propulsion systems, and controls are widely accepted in the aerospace industry for making critical design decisions, particularly in the preliminary design stage but increasingly for detailed design as well. There has been a very focused effort at developing numerical tools for analysis that are computationally expedient but do not sacrifice the fidelity required in the decision making process. More recently, there has been an increased awareness of accounting for uncertainty in the design process - this requires both the modeling of uncertainty in the analyses modules used in the simulation based design and, representing design failures in terms of reliability indices that can be related to probabilities of failure. Quantifying the effects of uncertainty on the optimal design configurations, and understanding its impact on the design decisions leads to the critical field of non-deterministic analysis and design. Reliability assessment and risk management methods are an integral part of the modern design process.

In a large-scale system design environment such as that involved in the development of an aerospace vehicle, the overall system is comprised of several subsystems or sub-disciplines. Designers in different disciplines need to interact and share results in order for the system design to progress towards a common optimal configuration. Multidisciplinary Design Optimization (MDO) refers to the collection of numerical and organizational tools that facilitates inter-disciplinary integration through optimization - allowing for the presence of uncertainties in analysis in any or all subsystems adds an additional degree of computational complexity in the design process.

The solution of such large, coupled design problems is often facilitated through a process of decomposition into sub-problems, each with its own set of design variables and constraints. Interactions exist between these sub-problems and must be managed during the design process. A considerable body of knowledge into effective approaches for decomposition has been developed, and must be revisited in the context of the presence of uncertainties. The presence of uncertainties also contributes to the necessity for reformulating the design optimization problem in terms of probabilistic measures of safety, and optimization algorithms best suited to solving such problems must be explored.

The focus of this paper is to provide a review of methods and formulations that efficiently allow the non-

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deterministic component to be integrated into an optimization framework - including specialized methods for multidisciplinary problems. Modeling of uncertainty and quantifying its effects is not a trivial task - when the underlying analyses behind the design process are computationally cumbersome, this assumes an additional degree of computational complexity. The remainder of the paper is organized as follows. Typical methods and formulations for solving MDO problems are first discussed, including how uncertainty analysis complicates these deterministic formulations. The quantification of uncertainty and the widely used approaches for solving nondeterministic design problems are then examined. Recent contributions in the area of non-deterministic analysis and optimization for multidisciplinary problems are discussed, including novel and promising research directions in the field of reliability and uncertainty modeling.

Multidisciplinary Design Optimization

The design of a complex engineering system involves many interacting components or parts. The interactions between the disciplines generally result from interactions between specific physical phenomena. In such an environment where everything appears to affect everything else, design of subsystems in isolation is clearly not the strategy of choice. Such types of systems are generally referred to as non-hierarchical systems, where there is no topology to a directed flow of information from one subsystem to another. In contrast, one can also identify systems where a distinct hierarchy between the system (S) and its subsystems (SS) is identified, and these are referred to as hierarchical systems (Fig. 1).

Linking of such coupled multidisciplinary analysis to a design optimization engine is impractical for a variety of reasons, including the following.

- The dimensionality of the design space may increase to a degree that obtaining reliable solutions to the optimization problem is rendered questionable.
- The ability to bring to bear human judgment on the quality of the solution as it progresses, is compromised in the presence of large number of variables and design constraints.
- For an iterative analysis environment that is typical of optimization, the presence of coupling between disciplines introduces an inner loop of iteration (analysis iteration) that adds very significantly to the computational costs.

Decomposition methods are introduced in multidisciplinary optimization to reduce large coupled optimization problems into a sequence of coordinated, smaller, more tractable subsystems. The subsystems not only allow for a reduction in problem dimensionality, but also allow for implementing specialized methods of analysis in each subsystem, and possibility of distributed, parallel processing. A general mathematical statement for the optimization problem can be written as follows.

Minimize \( F(X) \)

Subject to \( g_j(X) \leq 0 \quad j = l, m \)

\[ h_k(X) = 0 \quad k = l, p \]

\[ X^L \leq X \leq X^U \] (1)

Here \( X \) is the vector of design variables; superscripts ‘\( L \)’ and ‘\( U \)’ denote the lower and upper bounds, respectively; \( F(X) \) is the objective function that may be a scalar or a vector quantity and \( g_j(X) \) and \( h_k(X) \) are the inequality and equality constraints, respectively. In MDO problems, the dimensionality of the \( X \) vector can be high, and the computation of \( F(X), g_j(X) \) and \( h_k(X) \) may require analysis in many different coupled subsystems, a computationally cumbersome task.

The design problem described in eqn. (1) can be examined further by looking at a simplified non-hierarchical system consisting of two subsystems as shown in Fig. 2. This figure illustrates a situation in which the coupled multidisciplinary analysis in the two subsystems requires input variables \( x_1 \) and \( x_2 \), and coupling variables \( y_{12} \) and \( y_{21} \) computed in subsystems 1 and 2, respectively. The two subsystems produce output responses \( g_1 \) and \( g_2 \) that may be used directly in evaluating the objective and constraint functions. At the very outset, it is important to underscore the requirement for the underlying optimization formulation to be capable of handling distributed analysis codes. It can be impractical to build a single analysis routine to calculate all system responses, which the optimization engine can invoke. Under such distributed analysis architecture, a full system analysis typically requires the calculation of what are called coupling variables (the \( y \)’s in Fig. 2) in the MDO literature [1-4]. Thus, in order to obtain the values of responses \( g_1 \) and \( g_2 \) from the two subsystems at some value of the design variables, \( x \), the subsystem analyses need to be repeated several times (a fixed point iteration scheme), until a compatible set of coupling variable values is obtained.
When integrated into a design environment, the above analysis is linked to an iterative optimization algorithm that seeks queries on the objective and constraint function information for different combinations of the design variables. The iterative analysis involved with each such request of the optimizer is the principal contributor to the high computational cost. These demands on computational resources are further exacerbated by the need to provide sensitivity information for the objective and constraint functions.

Optimization Issues

Despite the multidisciplinary analysis being computationally expensive, there has been a significant amount of work in the integration of optimization algorithms within the multidisciplinary analysis shown above. A logical extension of the above discussion, then, would be to identify the combination of design variables $x^*$ that optimizes a performance measure dependent on the $x$'s and the $y$'s, and at the same time satisfies any disciplinary constraints.

An intuitively simple formulation often termed the all-in-one approach [1-4] (also termed multidisciplinary feasible method) can be used to solve the MDO problem. The formulation is given as,

$$\min_{x} J(x, y_{12}, y_{21})$$

subject to

$$s. t. \quad g_1(x_1, y_{21}) \leq 0$$
$$g_2(x_2, y_{12}) \leq 0$$

(2)

where $J$ represents the system objective and $g$ are the disciplinary constraints. Note that both the objective function and constraints are functions of the local design variables and the coupling variables, but only the design variables ($x$) are treated as the optimization variables. For each cycle of the optimization process, several complete multidisciplinary analyses (MDA) are required to obtain compatible values of the coupling variables.

An approach referred to as Simultaneous Analysis and Design (SAND) [1-4, 7,8] treats the coupling or linking variables ($y$) as optimization variables along with the design variables ($x$), and introduces the disciplinary analysis equations as equality constraints in the optimization.

$$\min_{x, y_{12}, y_{21}} J(x, y_{12}, y_{21})$$

subject to

$$s. t. \quad g_1(x_1, y_{21}) \leq 0$$
$$g_2(x_2, y_{12}) \leq 0$$
$$y_{12} = f_{12}(x_1, y_{21})$$
$$y_{21} = f_{21}(x_2, y_{12})$$

(3)

where the $f_i$ denote the respective disciplinary analyses. A distinct advantage of this approach is that the disciplinary analyses can be performed in parallel during every optimization cycle, in contrast with the series approach under the all-in-one formulation. The disadvantage is that unless both equality constraints are satisfied, the solution ($x, y$) is not compatible across all disciplines. As such, this approach is sometimes termed the Individual Discipline Feasible (IDF) approach [4].

Decomposition-based Design Optimization

The shortcomings of the all-in-one approach are evident from the description. Performing several full MDAs for each cycle of optimization can be prohibitively expensive, although this single-level formulation arguably leads to better convergence properties [2]. However, this formulation does not take any advantage of the fact that the original analysis has a distributed nature. In contrast, decomposition techniques for optimization break the problem into smaller optimization problems that are aligned with the distributed nature of the analysis. Each discipline solves its own local optimization problem, while a top-level solver (or optimizer) coordinates the values of the coupling variables to ensure compatibility. Depending on the architecture used, the decomposition may be categorized as hierarchical or non-hierarchical. Some well-known decomposition-based approaches include collaborative optimization, concurrent subsystem optimization [3, 4], among others. Collaborative optimization is a bi-level formulation where the top-level (system) optimizer optimizes a performance measure subject to compatibility constraints involving the coupling and shared variables. Each subsystem-level optimizer tries to minimize the discrepancy involving the coupling and shared variables, while satisfying local design constraints. Sobieszczanski-Sobieski and Haftka [4] present a more comprehensive review of decomposition-based MDO concepts.

Haftka and Watson [5] have proposed a modification to the CO-based approach in a formulation involving the use of constraint margin. In this method, every subsystem is provided with a budget/constraint margin and the asso-
ciated set of performance constraints. The subsystem level problem attempts to norm maximize this constraint margin while satisfying the performance requirements. The system level problem provides an upper bound to which the subsystem level problems should maximize the constraint margin. The presence of norm maximization at the subsystem level makes it non-smooth and excludes the use of standard derivative based solvers like SQP. Sakalkar and Hajela [6] overcome this drawback through the use of an extremum seeking function that replaces the maximum norm constraint at the subsystem level, with an equivalent envelope function for the constraint margin based CO approach. The system level optimization problem as written as follows, where $f_0$ is the system level objective function, $g_s$ are the system constraints and $\Omega_k$ is the extremum seeking cumulative constraint from the k-th subsystem.

$$\begin{align*}
\text{Min} & : f_0(X) \\
\text{s.t.} & : g_s(X) \leq 0 \quad S = 1 \ldots N \\
\Omega_k & \left[ g_k^{(i)}(X,y_k,p_k) \right] \leq 0 \\
X_l & \leq X \leq X_u
\end{align*}$$

The k-th subsystem level optimization problem is then written as follows.

$$\begin{align*}
\text{Min} & : \Omega_k \left[ g_k^{(j)}(X,y_k,p_k) \right] \\
\text{s.t.} & : g_k(X,y_k,p_k) = 0 \\
y_k^l & \leq y_k \leq y_k^u
\end{align*}$$

This approach has also been extended to include uncertainties in problem parameters and design variables in Ref. [6].

### Target Cascading

Target cascading [9, 10] is a relatively new approach to handling optimal design from a hierarchical systems perspective. In some ways, the target cascading approach resembles the collaborative optimization approach for decomposition based design, in that it attempts to solve individual optimization problems in parallel at the subsystem level, while maintaining inter-disciplinary compatibility through setting of targets at the system level. However, the target cascading approach is more geared towards hierarchical systems with multiple levels of design and analysis, whereas collaborative optimization is typically applied to problems with two levels of hierarchy: system and subsystem. Under target cascading, upper-level design responses and linking variable values are cascaded down to the lower level optimization problems (which may span different disciplines). The task of the lower level optimization is to reduce the discrepancy between the targets at the upper and lower level, and the linking variable values. The lower level optimization results can then be cascaded down to an even lower level, if needed. The next cycle requires the optimization information of the lower levels to be cascaded up to the system or super-system levels.

### Nondeterministic Modeling and Reliability Analysis

Before describing how uncertainty can be modeled and quantified in MDO systems, it is worthwhile to define some fundamental concepts used in the traditional probability theory approach, by far the most popular tool in non-deterministic analysis. Uncertain design parameters are most commonly modeled using Probability Distribution Functions (PDF’s), and we assume that enough statistical data is available to fit appropriate PDF’s for the random design variables. The first step in trying to obtain an estimate of system reliability is to define the response constraints in terms of the uncertain design parameters, $x$. The limit state function or the failure surface can then be defined as

$$g(x_1, x_2, \ldots, x_n) = 0$$

In the above equation, the combinations of the random variables for which $g(x) < 0$ indicate failure. The failure probability estimation problem of a structure, then, requires the solution to the following multidimensional integral,

$$p_f = \int \ldots \int f_\mathbf{x} (x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \ldots \, dx_n$$

where $f$ is the joint probability density function of the random design variables. The exact value of the above
integral is expensive to calculate, except for very simple functional forms. Monte Carlo sampling methods can be used to estimate this probability, but can be computationally expensive because the number of samples required is prohibitively high, especially for low values of the failure probability. First order approximation methods termed FORM (First Order Reliability Methods) are instead more commonly used to estimate this failure probability. These methods are based on the first order approximation [11,12] of the limit state. A reliability index $\beta$ is defined, which the optimal solution to the problem is,

$$\text{min } \beta = \sqrt{(x')^T (x')}$$

s.t. $g(x,x') = 0$  \hspace{1cm} (8)

where $x'$ is the set of original random variables transformed to the standard normal space. The point on the failure surface that defines the solution to the above optimization problem is termed the most probable failure point (MPP), and its location is used to quantify the failure probability, as

$$p_f = \Phi (-\beta )$$  \hspace{1cm} (9)

where $\Phi$ is the Cumulative Distribution Function (CDF) of the standard normal distribution. The above optimization formulation is usually termed the Reliability Index Approach (RIA). Some publications [8, 11, 13] mention that a slightly different approach termed Performance Measure approach (PMA) or inverse reliability index approach has more favorable computational properties. Under the PMA approach, the solution to the FORM problem is obtained for a pre-specified failure probability $p$. This formulation is especially suitable for problems where $g(x') = 0$ is not part of the solution space. The inverse reliability problem is given as

$$\text{min } g(x,x')$$

s.t. $\|x'\| = -\Phi^{-1} (p )$  \hspace{1cm} (10)

**Reliability Assessment for Multidisciplinary Systems**

Several recent publications [9, 10, 15-17] describe approaches for integrating reliability assessment within the multidisciplinary optimization framework. For the MDO example described earlier, if the design variables $x_1$ and $x_2$ are uncertain, the uncertainty propagates across the subsystems and to the responses $g_1$ and $g_2$. We need to evaluate the reliability of a particular design configuration $\{x^0\}$ with respect to the failure constraint $g_1$ that is part of subsystem 1.

**All-in-one Reliability Assessment Approach**

Perhaps the simplest way to perform reliability assessment would be to use an approach similar to the all-in-one MDO approach described above [7]. Clearly, for evaluating $g_1$ (and hence the reliability index for $g_1$), we need to know the value of the coupling variable $y_{12}$, which in turn would require the value of the coupling variable $y_{12}$ from subsystem 1. The all-in-one approach would then require a full system analysis for a single evaluation of $g_1$. The FORM-based formulation can then be given as

$$\text{min } \beta = \|x'\|$$

s.t. $g_1 (0, x', y_{21}) = 0$  \hspace{1cm} (11)

As an example, consider the following two-discipline coupled system. The disciplinary analyses and the constraint function equation are given as:

$$y_{12} = f_{12} (x,y_{21}) = x_1^2 + 2x_2 - x_3 + 2 \sqrt{y_{21}}$$

$$y_{21} = f_{21} (x,y_{12}) = x_1 x_4 + x_4^2 + x_5 + y_{12}$$

$$g_1 = -5 + x_1^2 + 2x_2 + x_3 + x_2 e^{-y_{21}} \leq 0$$  \hspace{1cm} (12)

To evaluate the constraint $g_1$ failure probability for the design configuration, \[0, 0, 0, 0, 0, 1, 2, 1, 1, 2\] we use the formulation in Eq. (6), where the constraint value $g_1$ is obtained by a fixed-point iteration for the disciplinary analyses. An SQP optimizer was used to solve the MPP search problem to obtain the final result as: $x^* = (-2.0272, -2.5426, -1.2713, 0, 0)$, which corresponds to a failure probability of $2.402 \times 10^{-4}$. The total number of full-system analysis provides an estimate of the computational expense, which in this case are 60. For more complex problems, the number of full system analysis may be much higher, and each system analysis could require several disciplinary analysis - a prohibitive computational cost in the presence of complex FEA and CFD models.
A top-level optimizer (Fig. 3) can be wrapped around the reliability assessment problem as follows.

\[
\min_x J(x, y_{12}, y_{21})
\]

s.t.

\[ P(g_1(x_1, y_{21}) > 0) \leq p_1 \tag{13} \]

In the above formulation (often termed Reliability-Based Design Optimization or RBDO), the quantity \( P(g_1(x_1, y_{21}) > 0) \leq p_1 \) constrains the failure probability of constraint \( g_1 \) to be less than equal to a preferred value \( p_1 \), and its value can be evaluated by the FORM-based multidisciplinary reliability assessment scheme described above. However, the Eq. (13) is now a multi-level optimization problem requiring several full system analyses to simply calculate the value of the probabilistic constraint involving \( g_1 \). As such, it comes with a high computational cost, and is not recommended even for smaller problems unless used in conjunction with additional efficiency enhancing tools.

**Computationally Efficient Tools for Uncertainty Modeling in MDO**

Generally speaking, computational efficiency of RBDO tools can be enhanced through more efficient solution methods for probabilistic constraint evaluation, efficient formulations for solving integrated nondeterministic optimization problems, and using approximations of analysis for ease of uncertainty based optimization. An important component in multidisciplinary reliability analysis scheme is obtaining the disciplinary probabilistic constraint - or the reliability assessment. In the previous section, we described the all-in-one approach for probabilistic constraint evaluation, which requires several full system analysis within an optimization loop. In this sub-section, we describe some recent formulations that overcome this computational inefficiency.

**Collaborative Reliability Assessment/ Unilevel Approach/ SAND Approach**

Du and Chen [7] present the collaborative reliability assessment approach for multidisciplinary systems, which has appeared in other publications under different names: unilevel approach by Agarwal et al. [8], single-loop approach by Chiralaksanakul and Mahadevan [11]. This formulation is on the lines of the SAND approach for MDO problems described earlier. The key feature of the new formulation is that the coupling variables \( (y) \) are made optimization variables along with the design variables \( x \). The disciplinary analysis then become equality constraints as part of the MPP search optimization problem. The formulation can then be written as

\[
\min_{x, y_{12}, y_{21}} \beta = \|x'\|
\]

s.t.

\[ g_1^0(x_1, x', y_{21}) = 0 \]

\[ y_{12} = f_{12}(x_1, y_{21}) \]

\[ y_{21} = f_{21}(x', y_{12}) \tag{14} \]

As can be seen, this formulation does not require any full system analysis, as the disciplinary analysis are now integrated within the MPP search problem. Note that the values of the coupling variables obtained at convergence will be in the transformed (or reduced) domain. The following example uses this formulation to estimate the reliability of a multidisciplinary system.

For the numerical problem described earlier, to obtain the failure probability for the constraint \( g_1 \) the optimization problem formulation is given as follows.

\[
\min_{x, y_{12}, y_{21}} \beta = \|x'\|
\]

s.t.

\[ -5 + x_1^2 + 2y_2 + x_3 + x_2 e^{-y_{21}} \leq 0 \]

\[ y_{12} = x_1^2 + 2x_2 + x_3 + 2 \sqrt{y_{21}} \]

\[ y_{21} = x_1 x_4 + x_4^2 + x_5 + y_{12} \tag{15} \]

In the above, the \( x' \) should be replaced with the transformation definition involving \( x \) and the mean and standard deviation of the design variable. The MPP obtained by solving the above problem using an SQP solver is \( x^{*} = [-2.0264, -2.5432, -1.2713, 0.0004, 0.0005] \) and the values of the coupling variables at the MPP are \( y_{12} = 10.9294 \) and \( y_{21} = 14.7269 \). As described above, this approach does not require any full system analysis, as the search towards the MPP is integrated into the IDF or SAND-like approach - both of them converging simulta-
Sequential Reliability Analysis for Multidisciplinary Systems

Ahn et al. [14] present another approach for reliability assessment of coupled multidisciplinary systems. Under this approach, the multidisciplinary analysis and the MPP search optimization are treated as two distinct (and unrelated) problems, and are performed iteratively in sequence. Beginning with a multidisciplinary analysis at $x'_k$ (k=1), one obtains the values of the coupling variables required for the evaluation of the constraint function $g$. The MDA is followed by a Newton update to the transformed design variables as part of the MPP search problem,

$$x'_{k-1} = \frac{\nabla_x g(x'_k,y'_k)^T x'_k - g(x'_k,y'_k)}{\| \nabla_x g(x'_k,y'_k) \|^2}$$

(16)

Where $y'_k$ are the coupling variable values obtained from a multidisciplinary analysis at the previous value of the iterate $x_k$. The new iterate $x'_{k+1}$ is fed back into the MDA to obtain a new set of coupling variable values. Although this approach feels different from what has been discussed so far, in principle, it is somewhat similar to the all-in-one reliability assessment approach - the difference being the Newton update that is employed for the MPP search instead of a regular optimizer. This Newton update requires the gradient information of the constraint equation $g$ with respect to the design variables $x$ evaluated at the current iterate $x'_k$. One must be careful while extracting first order information in a coupled system - the use of Global Sensitivity Equations (GSE) is recommended [4] for this application.

Sues et al. [16] present a similar reliability-based multidisciplinary optimization methodology. They incorporate an iterative process in which the design optimization phase and the MPP search phase are performed sequentially instead of in an integrated manner. The authors claim that this improves the computational efficiency of the optimization. Additional cost-saving measures employed are design variable screening and constraint screening, and fitting second order response surfaces for the objective function. The authors introduce the concept of single occurrence random variables and operational random variables. The focus seems to be on integrating reliability-based MDO tools with the available commercial analysis software, such as UG, PATRAN, and NASTRAN. The approach is implemented for an aircraft wing shape design and optimization.

Efficient Optimization-Integrated Reliability Assessment

In this subsection, we review efficient formulations for performing reliability-based design optimization in a multidisciplinary framework. The aim is to optimize some system objective, while simultaneously satisfying a bound on failure probabilities associated with one or more failure modes. An intuitive and simple way of doing this was described earlier, and is referred to as the nested loop or the double-loop approach [8], because the probabilistic constraint evaluation is an optimization problem embedded in an external optimization loop. The exact probability of failure is required for each cycle of the external optimization.

Unilevel Approaches

Unilevel approaches are a simple way to overcome the double-loop problem associated with the formulation in Eq. (13). The idea is to replace the inner optimization (MPP search problem) with its first order KKT optimality conditions, such that they are satisfied at the convergence of the outer optimization [8,11]. Doing so requires us to include the transformed variables $x'$ (which were part of the inner optimization) as optimization variables of the outer loop. The unilevel approach then takes the form

$$\min_{x,x'} J(x,y)$$

subject to

$$s \cdot t.$$ 

$$g_1(x,x',y) = 0$$

$$g_2(x,x',y) = 0$$

and so on.
\[ \nabla_x \cdot \left[ g_1(x, x', y) \right]^T x' + \| x' \| \| \nabla_x \cdot \left[ g_1(x, x', y) \right] \| = 0 \]

\[ \Phi \left( -\| x' \| \right) \leq p_1 \quad (17) \]

Note that the \( x' \) are also optimization variables, and they appear in the first two equality constraints which are the KKT optimality conditions for the MPP search. When this optimization problem converges, the \( x^* \) will represent the optimal design configuration with respect to the objective function \( J \), while \( x^* \) will represent the MPP because the KKT conditions for the MPP search optimization are satisfied. The last constraint ensures that the failure probability is less than the specified bound \( p_1 \).

For every cycle of iteration, however, one needs to perform a full system analysis to evaluate the objective and the constraint function values in the \( x \)-space, and the constraint function value in the standard normal space. To overcome this computational deficiency, a SAND-like approach discussed in Eq. (14) may be used. This requires inclusion of coupling variables \( y \) as optimization variables, in addition to the \( x \) and \( x' \). Furthermore, two copies of coupling variables are required: one associated with the original \( x \)-space, and another associated with the standard normal space [7, 17]. The formulation then takes the form

\[ \min_{x, x', y, y'} J(x, y) \]

s.t.

\[ g_1(x, x', y, y') = 0 \]

\[ \nabla_x \cdot \left[ g_1(x, x', y, y') \right]^T x' + \| x' \| \| \nabla_x \cdot \left[ g_1(x, x', y') \right] \| = 0 \]

\[ \Phi \left( -\| x' \| \right) \leq p_1 \quad (18) \]

If possible to accommodate the increase in design variables, the above formulation ensures that the exact probability of failure or an exact coupled analysis is not required for every cycle of iteration.

### Probabilistic Target Cascading

Recently, several researchers have used Analytical Target Cascading (ATC) to understand the propagation of uncertainty from the system to the subsystem level [18-20]. The original deterministic structure of the ATC remains unchanged for handling uncertainty. The only change is the inclusion of a probabilistic constraint in all of the sub-optimization problems, similar to the all-in-one approach described previously. However, the probabilistic constraint is evaluated during every cycle of iteration of each optimization module using an advanced mean value (AMV) method. The AMV approach does not solve the MPP optimization problem exactly, but requires a few evaluations of the function and gradient value of the constraint for estimating the probabilistic characteristics of the response.

### Response Surfaces

The third approach towards the goal of achieving computational efficiency in uncertainty modeling of multidisciplinary systems is that of using response surfaces. Known by different names in the design literature, such as surrogate models or met models, these are computationally benign mathematical representations of complex numerically intensive models (such as FEA, CFD) or of empirical observations. Advanced decision making tools, such as multiobjective optimization, reliability based design, and others can be directly coupled to complex numerical models through the use of metamodels. The typical met modeling problem statement is stated as follows: Given a set of data points \( x' \) \( (i = l, \ldots, n) \) and the corresponding simulation results, \( f'(x') \), \( i = l, \ldots, n \), construct a mathematical representation, \( f(x) \), that accurately represents the expensive simulation over the range of interest. The "metamodel" so obtained can be coupled with computational design meth-
ods, such as robust design, probabilistic design, and optimization.

The simplest and perhaps the most used response surface is the quadratic response surface, which is defined as

\[ f(x) = a_0 + \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \]  

(19)

where \( n \) is the number of parameters in the analysis, and the \( a_i \) are the parameters of the response surface model to be evaluated. Other non-parametric metamodeling techniques [21] include kriging, radial basis functions, neural networks, and support vector regression.

Response Surfaces in Uncertainty Based Design

A simple approach to getting the benefits of response surfaces in uncertainty analysis and design is to build the model first, performing disciplinary analyses at carefully selected sampling points, and using this model in lieu of exact analysis for reliability estimation. Another approach for using response surfaces is to build sequentially updating approximations (such as in a trust region approach) for solving the inner and/or optimization problems of a nested loop RBDO.

There are several examples in the literature where response surfaces have shown significant improvement in computational efficiency in reliability-based design optimization. Jin et al. [22, 23] follow the popular two-step procedure outlined above for a piston design problem: (i) creating kriging models for the responses of sound power level for piston slap noise and power loss due to piston friction, and (ii) using the metamodels for design optimization under uncertainty. They explore the use of Monte Carlo sampling using the original and the approximate models, and report a 94% reduction in CPU time compared to using the metamodels. Additionally, an interesting conclusion by the authors is that the Monte Carlo sampling applied to the original models tends to produce numerical noise in statistical quantities such as mean and variance of the response, leading to convergence problems for the subsequent optimization. In contrast, the same statistical quantities are noise-free if the corresponding metamodel is used - a characteristic feature of many approximation approaches in the numerical simulation domain. Fu and Jin [24] have reported accurate uncertainty-based design results using kriging metamodels for automotive simulations that require on average over 4 hours of CPU time for a single design sample. Mourelatos et al. [25] construct kriging metamodels using Latin Hypercube Sampling for the main bearing performance response of an IC engine. The metamodels obtained are integrated into a FORM approach for evaluating the probability characteristics. Use of response surfaces within an MDO framework is also common, and metamodels can be used to decouple the various disciplinary analyses. If this is achieved, any of the previously discussed RBDO formulations for multidisciplinary problems can be effectively used without concern for inter-disciplinary incompatibility. However, significant research is needed in the area of efficiently constructing metamodels for multidisciplinary systems.

Uncertainty Management in Multidisciplinary Systems

Several previous publications have emphasized the need to understand how uncertainty propagates across the various subsystems of a large-scale system design problem. Propagation of uncertainty in parameters across coupled subsystems is a complex phenomenon, and may not be easy to predict, unless one performs an expensive Monte Carlo analysis of the coupled analysis. Performing an uncertainty analysis in an MDO environment can be a highly computationally intensive task because the following computational elements are involved: (i) performing an optimization at the system level (ii) maintaining compatibility between subsystems through a distributed analysis framework, and (iii) satisfy probabilistic failure constraints in each subsystem.

Specifically, the issue of uncertainty management in coupled systems tries to address the question: "Can existing RBDO methods be used to predict and thereby modify the failure characteristics of individual subsystems, in an effort to manage the effects of uncertainty in individual subsystems?" The impact of uncertainty management can be significant. It can allow designers to decrease the effects of uncertainty in one subsystem at the cost of increasing such effects in another subsystem. Such a perspective could be especially important from the point of view of cost, where - in the event of failure - replacing one subsystem (or component) is significantly more expensive than some other subsystem. In such a case, the critical subsystem or failure mode can be made immune to the effects of uncertainty at the cost of another less critical subsystem, through design [17]. Mullur and Hajela [17] have proposed a SAND-like approach to manage uncertainty in...
coupled systems using multiobjective optimization concepts.

Gu et al. [26] present a framework for uncertainty propagation in coupled multidisciplinary systems. This work is presented from the perspective of robust design optimization and allows the estimation of variability in the output response of a multidisciplinary system. Particularly, the authors emphasize the worst-case uncertainty propagation through the subsystems, and its effect on the final response of interest. However, this approach does not provide the complete distribution information (say the CDF) for the response as a function of input parameter distribution, but it simply provides a range of variation for the response. This makes it useful for robust design applications, where the quantities of interest are typically performance and its variation. The authors use a multiobjective optimization approach (weighted sum) for optimizing the performance and minimizing variation. The variation also affects the constraints, whose feasibility is satisfied after the variation in the constraint response is included.

Smith and Mahadevan [27] present a probabilistic optimization approach for the design of aerospace vehicles. Optimization is performed at two levels: the global level associated with the geometry design for minimum weight, while the local design involves the design of a liquid hydrogen tank. Probabilistic constraints at both levels are evaluated using the all-in-one reliability assessment approach. Response surfaces are used for the expensive analyses. Shan and Wang [28] propose the concept of the failure surface frontier, which the authors claim overcomes the issues of accuracy of the FORM approach because of the first order approximation of the limit state function. The failure surface frontier is a hyper-surface consisting of non-dominated failure points in the limit state of a given failure region. The authors make use of kriging metamodels to construct the failure surface frontier, which is then used to efficiently assess the probability characteristics of given response.

Closing Remarks

The paper reviews efficient formulations for performing optimal design under uncertainty in the context of multidisciplinary systems. Issues of integrating reliability-based design optimization in the presence of coupled analysis are highlighted along with possible approaches to improve the computationally efficiency of the RBDO process. These include efficient reliability assessment approaches, use of efficient SAND-like analysis and design approaches, and use of computationally benign response surfaces. The RBDO process, if successfully implemented in a multidisciplinary framework, can significantly impact the outcome of the system design process. Advanced uncertainty management approaches can allow designers to preferentially distribute the adverse effects of uncertainty among subsystems most amenable to repair and to minimize product life cycle costs.

References


Fig. 1 A Coupled hierarchic system

Fig. 2 Coupled Multidisciplinary Analysis

Fig. 3 Integrated MDO and reliability analysis