REVISED ANALYTICAL CRITERIA FOR AIRCRAFT LATERAL-DIRECTIONAL INSTABILITIES

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Abstract

Following the recently revised presentation of the theory of small-perturbation flight dynamics by the authors, the present paper considers the newly corrected aircraft lateral-directional equations with trim angle of attack as a parameter. The dynamics is modeled by a set of second-order equations written in matrix form which makes it convenient to derive conditions for onset of instability. The static instability criterion is directly obtained and is seen to match entirely the condition for spiral mode instability. An approximate criterion for dynamic instability is obtained by considering the undamped lateral-directional dynamics, which happens to be a Hamiltonian system. The onset of dynamic instability corresponds to the occurrence of a Hamiltonian Hopf bifurcation which can be explicitly evaluated. This yields a new criterion for dynamic instability which is simpler and easier to use than the Routh discriminant. Also, the new criterion has the $C_{n\beta, \text{dyn}}$ parameter as a leading term which explains why this parameter may be used as an indicator for wing rock onset.

Introduction

Recently, the authors have presented a revised version of the flight dynamic theory of airplanes in longitudinal [1] and lateral-directional [2] flight. The dominant changes over the traditional presentation are as follows:

• The dynamic derivatives have been correctly redefined [3,4] as consisting of two components, one due to the relative angular velocity of the body with respect to the wind, and the other due to flow curvature.

• The timescales of relevance to the dynamics of the airplane as a rigid body in flight have been clearly stated [4,5] and the difference in order between them has been explicitly used to derive literal approximations to the dynamic modes.

• The static residual of the faster mode has been used [5,6] to derive improved approximations to the slower dynamic modes.

• The traditional notion of "static stability" about the three axes (pitch, roll and yaw) has been discarded in favor of a single, unified and correct definition of stability, common with other engineering and scientific disciplines [7,8].

The same changes are also reflected in the recent textbook [9] by the authors.

The issue of predicting the various instabilities encountered by an airplane in flight has been the focus of flight dynamics studies over the years [10]. The modern approach to this problem is to computationally locate points of instability through a bifurcation analysis [11,12]. For a realistic example of this approach, refer Figs.1-3, where the points of onset of instability (also called bifurcation points) are marked with open and filled squares. Each bifurcation point can be identified with a physical instability phenomenon. For instance, the point labeled ‘A’ in Fig.1 can be identified with spiral divergence, and point ‘D’ corresponds to a dutch roll instability leading to wing rock oscillations. However, one of the challenges is to predict the likelihood of occurrence of various instability phenomena during the selection of the airplane con-
figuration itself in order to nip the problem in the bud, so to speak. This requires reasonably simple analytical criteria that can be easily applied early in the aircraft design cycle, such as a criterion for spin susceptibility [13], for example.

Many airplanes, especially military ones, have been known to suffer from one form of lateral-directional instability or another (e.g., wing rock, nose slice, yaw departure, etc.) at low-to-moderate values of angle of attack. Fig. 4 shows the variation of the lateral-directional stability derivatives for an example fighter airplane for which a lateral-directional instability was observed just above 20 deg angle of attack [14]. The instability persisted for almost a further 10 deg of angle of attack until the trim capability of the horizontal tail was exhausted. From Fig. 5, the instability can be seen to roughly correlate with an unstable dutch roll mode and is approximately predicted by a parameter called \(C_{n\beta, \text{dyn}}\). Stability parameters, such as \(C_{n\beta, \text{dyn}}\), were usually obtained in an ad hoc manner and gained acceptance with mounting empirical evidence [15]. However, by all accounts, \(C_{n\beta, \text{dyn}}\) was a "static" instability parameter [16] and how it could predict wing rock onset, a "dynamic" instability phenomenon, remained a mystery. A formal derivation of instability criteria for lateral-directional flight dynamics, including the role of \(C_{n\beta, \text{dyn}}\), was put forward by Ananthakrishnan et al. [17] thereby settling the debate.

In light of the revised and corrected analysis of aircraft lateral-directional dynamics presented by the authors [2], it has become necessary to revisit the derivation of the lateral-directional instability criteria obtained earlier [17]. This paper rederives the analytical criteria for "static" and "dynamic" instability onset in aircraft lateral-directional dynamics.

**Corrected Lateral-Directional Dynamics**

Under the usual assumptions of small perturbations in the lateral-directional variables alone, about a straight and level flight trim state, the lateral-directional equations of motion may be written as [2]:

\[
\Delta \dot{\chi} = \left( \frac{g}{V^2} \right) \left( \frac{g S h}{W} \Delta C_y + \Delta \mu \right) \tag{1}
\]

\[
\Delta \dot{p_b} = \left( \frac{\bar{g} S h}{I_{xx}} \right) \Delta C_l \tag{2}
\]

\[
\Delta \dot{\nu}_b = \left( \frac{\bar{g} S h}{I_{zz}} \right) \Delta C_n \tag{3}
\]

where, \(p_b, \nu_b\) are the body-axis roll and yaw rates, respectively, \(\mu\) is the wind-axis (velocity vector) roll angle, \(\chi\) is the wind-axis yaw angle, \(\bar{g}\) is the dynamic pressure (= 1/2 \(\rho V^2\)), \(S\) is a reference area, usually the airplane wing planform area, \(b\) is the wing span, \(g\) is the acceleration due to gravity, \(W\) is the airplane weight, \(I_{xx}, I_{zz}\) are the moments of inertia, \(V^*\) is the trim velocity, and the perturbed aerodynamic coefficients are modeled as:

\[
\Delta C_r = C_{r\beta} \Delta \beta + C_{r \rho_1} (\Delta \rho_b - \Delta \rho_w) (b/2V^*) + C_{r \rho_2} \Delta \rho_w (b/2V^*) + C_{r \mu_1} (\Delta \mu - \Delta \mu_w) (b/2V^*) + C_{r \mu_2} \Delta \mu_w (b/2V^*) \tag{4}
\]

\[
\Delta C_c = C_{c\beta} \Delta \beta + C_{c \rho_1} (\Delta \rho_b - \Delta \rho_w) (b/2V^*) + C_{c \rho_2} \Delta \rho_w (b/2V^*) + C_{c \mu_1} (\Delta \mu - \Delta \mu_w) (b/2V^*) + C_{c \mu_2} \Delta \mu_w (b/2V^*) \tag{5}
\]

\[
\Delta C_n = C_{n\beta} \Delta \beta + C_{n \rho_1} (\Delta \rho_b - \Delta \rho_w) (b/2V^*) + C_{n \rho_2} \Delta \rho_w (b/2V^*) + C_{n \mu_1} (\Delta \mu - \Delta \mu_w) (b/2V^*) + C_{n \mu_2} \Delta \mu_w (b/2V^*) \tag{6}
\]

where \(\beta\) is the sideslip angle, the subscripts ‘\(b\)’ and ‘\(w\)’ refer to body-axis and wind-axis quantities respectively, and the various aerodynamic derivatives are defined as below:

\[
C_{r \beta} = \frac{\partial C_r}{\partial \beta} \bigg|_{\rho, \mu}; \quad C_{\beta \rho_1} = \frac{\partial C_r}{\partial \rho_b} \bigg|_{\beta, \mu}; \quad C_{r \beta} = \frac{\partial C_r}{\partial \beta} \bigg|_{\rho, \mu}; \quad C_{\beta \mu_1} = \frac{\partial C_r}{\partial \mu_w} \bigg|_{\beta, \rho_b};
\]

\[
C_{r \rho_1} = \frac{\partial C_r}{\partial (\rho_b - \rho_w)} (b/2V^*) \bigg|_{\beta, \mu}; \quad C_{\rho_1} = \frac{\partial C_r}{\partial (\rho_b - \rho_w)} (b/2V^*) \bigg|_{\beta, \mu}; \quad C_{\rho_2} = \frac{\partial C_r}{\partial \rho_w} (b/2V^*) \bigg|_{\beta, \mu};
\]

\[
C_{n \beta} = \frac{\partial C_n}{\partial \beta} \bigg|_{\rho, \mu}; \quad C_{\beta \rho_1} = \frac{\partial C_n}{\partial \rho_b} \bigg|_{\beta, \mu}; \quad C_{\beta \mu_1} = \frac{\partial C_n}{\partial \mu_w} \bigg|_{\beta, \rho_b};
\]

\[
C_{n \rho_1} = \frac{\partial C_n}{\partial (\rho_b - \rho_w)} (b/2V^*) \bigg|_{\beta, \mu}; \quad C_{\rho_1} = \frac{\partial C_n}{\partial (\rho_b - \rho_w)} (b/2V^*) \bigg|_{\beta, \mu}; \quad C_{\rho_2} = \frac{\partial C_n}{\partial \rho_w} (b/2V^*) \bigg|_{\beta, \mu};
\]

\[
C_{\beta \mu_2} = \frac{\partial C_n}{\partial \mu_w} (b/2V^*) \bigg|_{\beta, \rho_b}; \quad C_{\mu_1} = \frac{\partial C_n}{\partial \mu_w} (b/2V^*) \bigg|_{\beta, \rho_b}; \quad C_{\mu_2} = \frac{\partial C_n}{\partial \mu_w} (b/2V^*) \bigg|_{\beta, \rho_b};
\]
\[ C_{nr1} = \frac{\partial C_n}{\partial (r_w - r_w)} (b/2V) \] *;
\[ C_{nr2} = \frac{\partial C_n}{\partial r_w (b/2V)} \]

The ‘*’ indicates that the derivatives are to be evaluated at the trim state.

Note the distinct use of the ‘1’ and ‘2’ derivatives in Eqs.(4) to (6). The ‘1’ derivatives refer to the dynamic effect due to the relative angular rate between the body and the wind axes. That is, when the airplane is rotating relative to the wind. The ‘2’ derivatives represent what is called the flow curvature effect. That is, when the airplane is flying along a curved flight path, the body and wind axis angular rates are identical. For instance, an arbitrary body-axis yaw rate \( \Delta \theta_w \) can be split into two components: one equal to the wind-axis yaw rate \( \Delta \theta^w \), and another equal to the difference between the two, \( (\Delta \theta^w - \Delta \theta^b) \). The first, \( \Delta \theta^w \), multiplies the ‘2’ derivative, and the second, \( (\Delta \theta^w - \Delta \theta^b) \), multiplies the ‘1’ derivative.

We have assumed small perturbations for the lateral-directional derivatives but the trim angle of attack \( \alpha^* \) is not limited to be small since it is required to be varied to adjudge the point of onset of instability of the modes. However note that the trim flight path angle is always zero for level flight, i.e., \( \gamma^* = 0 \).

We have the following relations between the perturbed body- and wind-axis roll and yaw rates [4]:
\[ \Delta p_w - \Delta p_w = \Delta \beta \sin \alpha^* \quad \text{and} \quad \Delta r_w - \Delta r_w = -\Delta \beta \cos \alpha^* \]

\[ \Delta p_w = \Delta \mu - \Delta \chi \sin \gamma^* = \Delta \mu \quad \text{and} \]
\[ \Delta r_w = \Delta \chi \cos \gamma^* \cos \Delta \mu = \Delta \chi \]

Using these relations, the aerodynamic model in Eqs.(4) to (6) can be written as:

\[ \Delta C = C_{\beta \alpha} \Delta \beta + C_{\beta \mu} \Delta \mu \sin \alpha^* (b/2V^*) + C_{\mu \beta} \Delta \mu (b/2V^*) + C_{\mu \alpha} \Delta \mu (b/2V^*) \]

\[ \Delta C_{\beta} = C_{\beta \alpha} \Delta \beta + C_{\beta \mu} \Delta \mu \sin \alpha^* (b/2V^*) + C_{\mu \beta} \Delta \mu (b/2V^*) + C_{\mu \alpha} \Delta \mu (b/2V^*) \]

\[ \Delta C_{\mu} = C_{\mu \mu} \Delta \mu + C_{\mu \beta} \Delta \beta \sin \alpha^* (b/2V^*) + C_{\mu \chi} \Delta \mu (b/2V^*) + C_{\mu \alpha} \Delta \mu (b/2V^*) \]

Now we can insert the aerodynamic model of Eqs.(10) through (12) in the lateral-directional Eqs. (1) through (3) to give the complete set of equations as below:

\[ \Delta \dot{\chi} = \frac{q}{p} \left( \frac{\hat{S} S}{W} \right) \left( C_{p \beta} \Delta \beta + C_{p \mu} \Delta \mu \sin \alpha^* (b/2V^*) + C_{p \chi} \Delta \mu (b/2V^*) \right) + \Delta \mu \]

\[ \Delta \dot{\mu} = \frac{q}{T_{\alpha \beta}} \left[ C_{p \beta} \Delta \beta + C_{p \mu} \Delta \mu \sin \alpha^* (b/2V^*) + C_{p \chi} \Delta \mu (b/2V^*) \right] \]

\[ \Delta \dot{\beta} = \frac{q}{T_{\alpha \beta}} \left[ C_{p \beta} \Delta \beta + C_{p \mu} \Delta \mu \sin \alpha^* (b/2V^*) + C_{p \chi} \Delta \mu (b/2V^*) \right] \]

Lateral-Directional Equations in Second-order Form

We next write Eqs.(13) through (15) as a set of two second-order differential equations. First of all, the rate derivatives of the side force coefficient, \( C_{p \beta} \), \( C_{p \mu} \), \( C_{p \beta} \), \( C_{p \chi} \), are usually of lesser importance and may be ignored. Then, Eq.(13) reduces to:

\[ \Delta \dot{\chi} = \frac{q}{p} \left( \frac{\hat{S} S}{W} \right) C_{p \beta} \Delta \beta + \Delta \mu \]

Before proceeding further, for ease of algebraic manipulation, let us define some short symbols:

\[ \hat{S} = Y \beta, \quad \frac{\hat{S} \hat{S} b}{T_{\alpha \beta}}, \quad \frac{\hat{S} \hat{S} b}{T_{\alpha \beta}} \]

\[ C_{p \beta} = \frac{\hat{S} S}{W} \]

\[ C_{p \mu} = \frac{N}{p} \]

\[ C_{p \chi} = \frac{\hat{S} \hat{S} b}{T_{\alpha \beta}} \]

\[ C_{p \beta} = \frac{\hat{S} \hat{S} b}{T_{\alpha \beta}} \]
\[
\frac{q S h}{I_{zz}} C_{br2} \left( \frac{b}{2V} \right) = N_{p2} : \left( \frac{q S b}{I_{zz}} \right) C_{rr1} \left( \frac{h}{2V} \right) = N_{r1} ;
\]
\[
\frac{q S b}{I_{zz}} C_{br2} \left( \frac{b}{2V} \right) = N_{r2} : \left( \frac{q S b}{I_{xx}} \right) C_{br2} \left( \frac{b}{2V} \right) = L_{p2} ;
\]
\[
\frac{q S b}{I_{xx}} C_{br1} \left( \frac{h}{2V} \right) = L_{r1} : \left( \frac{q S b}{I_{xx}} \right) C_{br2} \left( \frac{b}{2V} \right) = L_{r2} ;
\]
\[
\frac{q S b}{I_{xx}} C_{br1} \left( \frac{h}{2V} \right) = N_{p1} : \left( \frac{q S b}{I_{xx}} \right) C_{br1} \left( \frac{h}{2V} \right) = L_{p1} (17)
\]
So, Eq.(16) can be compactly written as
\[
\Delta \dot{\chi} = \frac{g}{V} \left[ Y_{up} \Delta \dot{\beta} + \Delta \mu \right] (18)
\]
The perturbed wind-axis Euler angle rates can be related to the body-axis angular rates as follows [4]:
\[
\Delta \dot{\mu} = \Delta p_{b} \cos \alpha^{*} + \Delta r_{b} \sin \alpha^{*} \quad \text{and}
\]
\[
\Delta \dot{\chi} = - \Delta p_{b} \sin \alpha^{*} + \Delta r_{b} \cos \alpha^{*} + \Delta \ddot{\beta} \quad (19)
\]
Differentiating the first of Eq.(19) once with respect to time, and using Eqs.(14) and (15) on the right-hand side with the short forms of Eq.(17) gives the following equation for roll:
\[
\Delta \ddot{\mu} = L_{p} \dot{\beta} + L_{p1} \Delta \dot{\beta} \sin \alpha^{*} + L_{r2} \Delta \ddot{\chi} \quad (20)
\]
Where the ‘primed’ short symbols are defined in the following manner:
\[
L_{(\_)}^{\prime} = L_{(\_)} \cos \alpha^{*} + N_{(\_)} \sin \alpha^{*} \quad (21)
\]
Using \( \Delta \ddot{\chi} \) from Eq.(18), we can complete Eq.(20) as below:
\[
\Delta \ddot{\mu} = \left[ L_{p} \dot{\beta} + \left( \frac{g}{V} \right) Y_{up} \right] \Delta \beta + L_{p1} \Delta \ddot{\beta} \sin \alpha^{*} + L_{r2} \Delta \ddot{\chi} \quad (22)
\]
Which on collecting like terms, appears as:
\[
\Delta \ddot{\mu} = \left[ L_{p} \dot{\beta} + \left( \frac{g}{V} \right) Y_{up} \right] \Delta \beta + L_{p1} \dot{\beta} \sin \alpha^{*} + L_{p2} \Delta \mu = 0 \quad (23)
\]
Similarly, differentiating the second of Eq.(19) and using Eqs.(14) and (15) yields:
\[
\Delta \dddot{\chi} = N_{p1} \dot{\beta} + N_{p1} \Delta \dot{\beta} \sin \alpha^{*} + N_{r2} \Delta \ddot{\mu}
\]
\[
+ N_{r1} (- \dot{\beta}) \cos \alpha^{*} + N_{r2} \Delta \dot{\chi} + \Delta \dddot{\beta} \quad (24)
\]
The primed short symbols in Eq.(24) are defined as below:
\[
N_{(\_)}^{\prime} = - L_{(\_)} \sin \alpha^{*} + N_{(\_)} \cos \alpha^{*} \quad (25)
\]
Again replacing \( \Delta \ddot{\chi} \) for its derivative from Eq.(18), and collecting like terms in Eq.(24) gives the following equation for yaw:
\[
\Delta \ddot{\beta} = \left[ N_{p1} \dot{\beta} + N_{p1} \Delta \dot{\beta} \sin \alpha^{*} + N_{r2} \Delta \ddot{\mu} \right]
\]
\[
\left[ N_{r1} (- \dot{\beta}) \cos \alpha^{*} + N_{r2} \Delta \dot{\chi} + \Delta \dddot{\beta} \right] \quad (26)
\]
Equations (23) and (26) form the set of two second-order lateral-directional small-perturbation equations. For convenience, these are collected together and presented in Table-1.

It can be verified that these two equations of second order in the variables \( \Delta \mu \) and \( \Delta \beta \) are coupled because of the \( \Delta \ddot{\beta} \) terms in the \( \Delta \beta \) (rolling moment) equation, and the \( \Delta \mu \) terms in the \( \Delta \beta \) (pitching moment) equation. Note that that \( \Delta \chi \) (side force) equation has been absorbed into Eqs.(23) and (26); hence there is no separate equation for \( \Delta \chi \). In fact, the variable \( \Delta \chi \) has itself been eliminated.

**Lateral-Directional Equations in Matrix Form**

For further analysis, Eqs.(23) and (26) need to be cast in matrix form, as below:

\[
\begin{bmatrix}
\Delta \ddot{\mu} \\
\Delta \ddot{\beta}
\end{bmatrix} =
\begin{bmatrix}
L_{p} \dot{\beta} + \left( \frac{g}{V} \right) Y_{up} \\
N_{p1} \dot{\beta} + N_{p1} \Delta \dot{\beta} \sin \alpha^{*} + N_{r2} \Delta \ddot{\mu} \right]
\end{bmatrix}
\begin{bmatrix}
\Delta \beta \\
\Delta \dot{\beta}
\end{bmatrix} + L_{p2} \Delta \mu (23)
\]

\[
\begin{bmatrix}
\Delta \ddot{\mu} \\
\Delta \ddot{\beta}
\end{bmatrix} =
\begin{bmatrix}
L_{p} \dot{\beta} + \left( \frac{g}{V} \right) Y_{up} \\
N_{p1} \dot{\beta} + N_{p1} \Delta \dot{\beta} \sin \alpha^{*} + N_{r2} \Delta \ddot{\mu} \right]
\end{bmatrix}
\begin{bmatrix}
\Delta \beta \\
\Delta \dot{\beta}
\end{bmatrix} + L_{r2} \Delta \ddot{\chi} (24)
\]
The main entries in the stiffness matrix are the so-called damping terms -
stiffness matrix. Previously [17], due to the incorrect aerodynamic model, the damping terms used to appear in the stiffness matrix, whereas the flow curvature terms due to wind-axis yaw rate - appear in the damping matrix, whereas the flow curvature terms with respect to roll - do not yield a simple expression that can be usefully
determined.

The criterion for static instability is expected to correspond to the spiral mode
eigenvalue previously. Similar to the case of static instability, there is an exact
instability criterion in Eq.(29) and the spiral mode eigenvalue previously. The connection between the static
instability criterion in Eq.(29) and the spiral mode eigenvalue further reinforces this point -
static instability, cannot be dependent on "damping" terms, such as . It is the flaw in the traditional
aerodynamic model that was responsible for damping terms such as .

Table-1 : Summary of Lateral-directional Dynamics Equations

<table>
<thead>
<tr>
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<th>Yaw Dynamics</th>
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|                  | \( \Delta \beta + \left[ N_{r1} \cos \alpha - N_{p1} \sin \alpha + \left( \frac{g}{V} \right) Y_\theta N_{r2} \right] \Delta \beta - \left[ N_{1r} \cos \alpha' - N_{p1} \sin \alpha' + \left( \frac{g}{V} \right) Y_\theta \right] \Delta \beta \)
|                  | + \( \left( \frac{g}{V} \right) N_{r2} \Delta \mu + \left[ N_{p2} - \left( \frac{g}{V} \right) \right] \Delta \mu = 0 \) |

\[
\begin{align*}
\begin{bmatrix}
\Delta \beta \\
\Delta \mu
\end{bmatrix} & + \begin{bmatrix}
N_{r1} \cos \alpha - N_{p1} \sin \alpha + \left( \frac{g}{V} \right) Y_\theta \\
L_{r1} \cos \alpha - L_{p1} \sin \alpha
\end{bmatrix}
\begin{bmatrix}
\Delta \beta \\
\Delta \mu
\end{bmatrix} \\
& + \begin{bmatrix}
N_{p2} - \left( \frac{g}{V} \right) \\
L_{p2} - \left( \frac{g}{V} \right)
\end{bmatrix}
\begin{bmatrix}
\Delta \beta \\
\Delta \mu
\end{bmatrix} = 0
\end{align*}
\]

(27)

Which can be briefly represented by the following matrix equation:

\[
\ddot{y}_{lat} + C_{lat} \dot{y}_{lat} + K_{lat} y_{lat} = 0
\]

(28)

Where

\[
y_{lat} = \begin{bmatrix}
\Delta \beta \\
\Delta \mu
\end{bmatrix}
\]

And \( C_{lat} \) , \( K_{lat} \) are the damping and stiffness matrix, respectively, from Eq.(27). Note that the prominent entries in the damping matrix are the so-called damping terms - \( N_{r1} \) , \( L_{r1} \) , \( N_{p1} \) , \( L_{p1} \) , and these do not show up in the stiffness matrix. Previously [17], due to the incorrect aerodynamic model, the damping terms used to appear in the stiffness matrix as well, which is obviously erroneous. The main entries in the stiffness matrix are \( N_{r2} \) , \( L_{r2} \) . The flow curvature effect terms with respect to roll - \( L_{p2} \) , \( N_{p2} \) - appear in the damping matrix, whereas the flow curvature terms due to wind-axis yaw rate - \( N_{r2} \) , \( L_{r2} \) - are seen in the stiffness matrix.

Static Instability Criterion

The criterion for static instability is given by \( d et (K_{lat}) = 0 \) , which is equivalent to the Routh criterion requiring the constant term in the characteristic polynomial to be zero. This condition, labeled \( S_{lat} \), can be easily evaluated from Eq.(27) to be:

\[
S_{lat} : \left( \frac{g}{V} \right) \left[ L_{p2} N_{r2} - N_{p2} L_{r2} \right] = 0
\]

(29)

The expression on the left-hand side of Eq.(29) can be verified to be identical to the formula for the spiral mode
eigenvalue [2], which makes perfect sense since the onset of static instability is expected to correspond to the spiral
mode pole being at the origin of the complex plane. It has already been explained [2] why it is reasonable to have the
flow curvature terms, \( N_{r2} \) , \( L_{r2} \) , in the expression for the spiral mode eigenvalue. The connection between the static
instability criterion in Eq.(29) and the spiral mode eigenvalue further reinforces this point - spiral mode instability,
a static instability, cannot be dependent on "damping" terms, such as . It is the flaw in the traditional
aerodynamic model that was responsible for damping terms such as .

Hamiltonian Approximation of Lateral-Directional Equations

Similar to the case of static instability, there is an exact criterion for the oscillatory or "dynamic" instability given
by the Routh discriminant [18]; however that criterion does not yield a simple expression that can be usefully
employed. Hence, the search for an approximate dynamic instability criterion.

Typically, at moderate values of angle of attack $\alpha^*$, the eigenvalues of the lateral-directional model in Eq.(27) turn out to be a complex pair of lightly damped poles representing the dutch roll mode and a second complex pair of low-damped poles, called the lateral phugoid, formed by the merger of the roll (rate) and spiral eigenvalues. Thus, the lateral-directional dynamics as a whole is poorly damped around the point of onset of the dynamic instability. Therefore, one may justifiably approximate the lateral-directional dynamics as an undamped system.

In the traditional way of presenting the lateral-directional dynamic equations, it is quite impossible to identify which terms represent the "damping" and to isolate them in order to arrive at an undamped approximation. However, by transforming the lateral-directional dynamics into second-order equations and presenting them in a matrix form as in Eq.(27), the damping terms are clearly separated. By setting the supposedly negligible $C_{lat}$ matrix to zero, one can obtain an undamped approximation to the lateral-directional dynamics as follows:

$$
\begin{bmatrix}
\Delta \beta^i \\
\Delta \mu^i
\end{bmatrix} =
\begin{bmatrix}
N_i' + \left( \frac{g}{V} \right)_i Y_i \beta N_i \\
- \left( \frac{g}{V} \right)_r L_i - \left( \frac{g}{V} \right)_r L_{i2}
\end{bmatrix}
\begin{bmatrix}
\Delta \beta \\
\Delta \mu
\end{bmatrix} = 0
$$

(30)

Which is equivalently represented by,

$$
\ddot{Y}_{lat} + K_{lat} Y_{lat} = 0
$$

(31)

This manner of approximating the aircraft small-perturbation dynamics as a Hamiltonian system was first introduced by Ananthkrishnan et al. [17]. The onset of dynamic instability in the damped lateral-directional dynamics Eq.(27) can then be approximated by the dynamic instability of the undamped (Hamiltonian) system in Eq.(30).

**Dynamic Instability Mechanism in Hamiltonian Systems**

The eigenvalues of a Hamiltonian system are constrained to be symmetric about the origin in the complex plane. This means that only four possible arrangements of eigenvalues can occur:

- Two pairs of complex conjugate eigenvalues on the imaginary axis; $\pm i \omega_1 , \pm i \omega_2$
- Two pairs of real eigenvalues; $\pm \sigma_1 , \pm \sigma_2$
- One pair of complex conjugate eigenvalues on the imaginary axis, another pair of real eigenvalues; $\pm i \omega, \pm \sigma$
- Two pairs of complex conjugate eigenvalues; $\pm (\sigma \pm i \omega)$, $\sigma \neq 0$

Of these, only the first option indicates stable behavior; the other three represent various forms of instability. When the eigenvalues in the first option move along the imaginary axis towards each other with a varying parameter (in our case, the trim angle of attack $\alpha^*$) and collide, they can branch out as a pair of complex conjugate eigenvalues (mirror image of each other) as in the fourth option. This is indicated graphically in Fig.6 where the arrows indicate the direction of movement of the eigenvalues with varying parameter. This is the dynamic instability mechanism in Hamiltonian systems and the onset of dynamic instability occurs at the point where the eigenvalues collide and sit atop each other on the imaginary axis before separating into each half-plane. Technically, this is also known as a Hamiltonian Hopf bifurcation mechanism [19].

**Dynamic Instability Criterion**

The condition for the onset of dynamic instability in the system of Eq.(31) can be given by the discriminant of the following quadratic equation in $\lambda^2$ being zero:

$$
| \lambda^2 I + K_{lat} | = 0
$$

(32)

Or equivalently, by defining $\lambda^2 = \tau$, by the discriminant of the following quadratic equation in $\tau$ being zero:

$$
| \tau I + K_{lat} | = 0
$$

(33)

Which may be explicitly written down as follows:

$$
\begin{bmatrix}
\tau + \left[ N_i' + \left( \frac{g}{V} \right)_i Y_i \beta N_i \right] & \left( \frac{g}{V} \right)_i N_i \\
- \left[ L_i' + \left( \frac{g}{V} \right)_i Y_i \beta L_i \right] & \tau - \left( \frac{g}{V} \right)_r L_{i2}
\end{bmatrix} = 0
$$

(34)
Expanding Eq.(34), we get,
\[ \tau^2 + \left[ N_{\beta}' + \frac{g}{V^*} Y_{\beta} N_{r2}' - \frac{g}{V^*} L_{r2}' \right] \tau = 0 \]
\[ \tau = \left[ \frac{g}{V^*} \right] \left( L_{\beta}' - N_{\beta}' \right) \] \[ \text{Eq.}(36) \]

Where the last term with underbraces is the same as \( S_{lat} \) in Eq.(29). The condition that the discriminant be zero gives the dynamic instability criterion:
\[ D_{lat} \cdot \left[ N_{\beta}' + \left( \frac{g}{V^*} \right) Y_{\beta} N_{r2}' - \left( \frac{g}{V^*} \right) L_{r2}' \right] = 0 \]
\[ \text{Eq.}(36) \]

\( D_{lat} \cdot \text{Eq.}(36) \) gives the approximate condition for onset of oscillatory instability in the lateral-directional dynamics, which usually manifests as the outcome of an unstable dutch roll dynamics called wing rock.

A summary of the static and dynamic instability criteria in lateral-directional dynamics is presented in Table-2.

Note that the expression in Eq.(36) is different from a similar one derived earlier [17] insofar as the flow curvature derivatives \( N_{r2}' \), \( L_{r2}' \) appear in Eq.(36) as against the damping derivatives \( N_{\beta}' \), \( L_{\beta}' \) that erroneously appeared earlier [17] due to the traditional aerodynamic model in use then being incorrect.

The leading term in the \( D_{lat} \) criterion is \( N_{\beta}' \), which using Eq.(25) expands as,
\[ N_{\beta}' = -L_{\beta} \sin \alpha^* + N_{\beta} \cos \alpha^* \] \[ \text{Eq.}(37) \]

And on using the full form of the symbols from Eq.(17), we can write Eq.(37) in terms of the aerodynamic derivatives as,
\[ \frac{g S \beta}{I_{zz}} C_{n\beta} \cos \alpha^* - \frac{g S h}{I_{xx}} C_{n\beta} \sin \alpha^* \]
\[ = \frac{C_{n, \beta, \text{dyn}}}{C_{n, \text{beta}, \text{dyn}}} \]
\[ \text{Eq.}(38) \]

Where, as indicated, the term with the underbrace is usually called \( C_{n, \beta, \text{dyn}} \). It is in this manner that \( C_{n, \beta, \text{dyn}} \) becomes one of the factors but not the sole factor indicative of wing rock onset. For example, in Fig.5 both \( C_{n, \beta} \) and \( C_{n, \beta, \text{dyn}} \) cross zero close to the point of onset of instability, but for another airplane in Fig.7, even though \( C_{n, \beta} \) crosses zero, \( C_{n, \beta, \text{dyn}} \) remains positive and lateral-directional instability is not observed. Thus, over the years, \( C_{n, \beta} \) has gained credibility as an indicator for the likelihood of wing onset, but there was no cogent explanation for why this could be so. The work by Ananthkrisnan et al. [17] reinforced by the present paper has placed the use of \( C_{n, \beta, \text{dyn}} \) as a lateral-directional instability parameter on a firm theoretical footing. However, it must be noted that the entire criterion \( D_{lat} \) in Eq.(36) is the correct indicator for wing rock onset, not just \( C_{n, \beta, \text{dyn}} \) alone.

**Conclusion**

The present paper has derived criteria for onset of static and dynamic instability in airplane lateral-directional dynamics - an exact criterion \( S_{lat} \) for static instability and an approximate criterion \( D_{lat} \) for dynamic instability. Both these criteria are improved and corrected versions of the instability onset conditions first obtained by Ananthkrisnan et al.[17]. Notably, it is no longer the so-called "damp-

<table>
<thead>
<tr>
<th>Table-2 : Summary of Lateral-directional Instability Criteria</th>
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<tbody>
<tr>
<td><strong>Static Instability</strong></td>
</tr>
<tr>
<td>[ S_{lat} : \left( \frac{g}{V^*} \right) \left( L_{\beta}' N_{r2}' - N_{\beta}' L_{r2}' \right) = 0 ]</td>
</tr>
<tr>
<td><strong>Dynamic Instability</strong></td>
</tr>
<tr>
<td>[ D_{lat} : \left[ \left( N_{\beta}' + \left( \frac{g}{V^<em>} \right) Y_{\beta} N_{r2}' \right) \left( \frac{g}{V^</em>} \right) L_{r2}' \right] = 0 ]</td>
</tr>
</tbody>
</table>

\[ S_{lat} \]
ing” derivatives, \(N_{r1}', L_{r1}'\), that feature in these criteria but the corresponding “flow curvature” derivatives, \(N_{r2}', L_{r2}'\). This is a consequence of the corrected aerodynamic model presented by the authors in their recent works \([1,2]\). As a result, onset of spiral instability (obtained from \(S_{lat}\)) and wing rock onset (given by \(D_{lat}\)) predicted by the criterion derived in this paper may be expected to match more closely with that observed in flight tests.

References


Fig.1 Typical Bifurcation Diagram Showing Thrust Vs. Mach Number. The open and filled squares are bifurcation points (points of onset of instability)

Fig.2 Typical Bifurcation Diagram Showing the Longitudinal Variables, Angle of Attack, Pitch Angle, Flight Path Angle, and Body Pitch Rate. The open and filled squares are bifurcation points (points of onset of instability)

Fig.3 Typical Bifurcation Diagram Showing the Lateral-directional Variables, Body Roll Rate, Body Yaw Rate, Sideslip Angle, and Body Roll Angle. The open and filled squares are bifurcation points (points of onset of instability)

Fig.4 Lateral-directional Stability Derivatives for an Example Airplane (from Ref.14)
Fig. 5 Lateral-directional Instability Onset for an Example Airplane (from Ref. 14)

Fig. 6 Dynamical Instability Mechanism in Hamiltonian Systems

Fig. 7 Variation of Lateral-directional Stability Derivatives for Another Fighter Airplane (from Ref. 20)