USE OF BIFURCATION AND CONTINUATION METHODS FOR AIRCRAFT TRIM AND STABILITY ANALYSIS - A STATE-OF-THE-ART

Aditya Paranjape*, Nandan Kumar Sinha** and Narayan Ananthkrishnan+

Abstract

The bifurcation and continuation methodology has evolved over the last two decades into a powerful tool for the analysis of trim and stability problems in aircraft flight dynamics. Over the years, bifurcation methods have been employed to deal with a variety of problems in aircraft dynamics, such as predicting high angle of attack behavior, especially spin, and studying instabilities in rolling maneuvers. The bifurcation methodology has served as a tool for the design of flight control systems, and is promising to be a useful tool in the aircraft design, simulation, testing, and evaluation process. In the present paper, we describe the state-of-the-art in the use of bifurcation and continuation methods for the analysis of aircraft trim and stability with a few illustrative examples. Both the standard and extended bifurcation analysis procedures are discussed and typical results for instabilities in high-\(\alpha\) flight and inertia-coupled roll maneuvers are shown. This is followed by several problems in nonlinear flight dynamics where bifurcation and continuation methods have been fruitfully applied to yield effective solutions. Finally, the use of bifurcation theory to arrive at analytical instability criteria is demonstrated for the aircraft roll coupling and wing rock problems. 78 references have been cited in the text.

Introduction

Nonlinear problems in aircraft flight dynamics and control have been well recognized and widely documented ever since the dawn of aviation [1]. However, in the early days, the prediction and solution of these problems, especially in the design phase, was seriously limited by lack of analytical tools and poor simulation capabilities. Among the earliest nonlinear problems of aircraft stability and control to draw serious attention was the issue of roll-coupled or inertia-coupled flight dynamics, first predicted by Phillips [55]. A closely-related problem of roll-yaw coupling in missile flight dynamics was described by Nicolaides [50]. Reviews of early work on these problems have been provided by Hacker and Oprisiu [30], and Murphy [48], respectively; for a more recent summary of the roll-coupling problem, see Jahnke [34]. Initially, approximate analytical methods were used to predict conditions for onset of instability, but these were not always reliable, as will be shown in this paper for the roll-coupling problem. Later, as computing capabilities improved, greater reliance was placed on numerical simulations to predict instability onset as well as post-instability nonlinear dynamical behavior. Unfortunately, the sheer number of simulation runs required to locate and identify a problem made it very expensive and time consuming. Advances in fighter aircraft technology and combat requirements in the past three decades or so have made it necessary for airplanes to operate beyond the limits of linear aerodynamics [32]. This has brought in new nonlinear problems due to flight at high angles of attack and in critical flight regimes, typically under post-stall conditions [51]. Recent demands on aircraft agility and performance, such as that for the X-31 enhanced fighter maneuverability (EFM) demonstrator [2], have pushed the envelop even further, making modern airplanes susceptible to a wide range of stability and control problems [52].

A key development in the field was the introduction of bifurcation and continuation methods to problems in flight dynamics by Carroll and Mehr [15], Guicheteau [28], and

* Graduate Student, Department of Aerospace Engineering, Indian Institute of Technology Bombay, Powai, Mumbai-400 076, India, Email : adityap@iitbombay.org; Presently in Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA

** Assistant Professor, Department of Aerospace Engineering, Indian Institute of Technology Madras, Chennai-600 036, India, Email : nandan@ae.iitm.ac.in

+ Director, Coral Digital Technologies Pvt Ltd, Bangalore-560 043, India, Email : dr.akn.19@gmail.com

Manuscript received on 03 Jul 2007; Paper reviewed, revised and accepted on 22 Feb 2007
Zagaynov and Goman [78]. The use of continuation algorithms made it possible to smartly compute an entire family of steady state (trim) solutions for varying values of a control parameter, for example, elevator deflection. Using numerical differentiation, it was possible to obtain the linearized dynamics at each trim state, and hence compute the stability of each trim state. Coupled with knowledge from bifurcation theory, points of onset of instability (or departure) and the nature of the departed solution could be identified. Modern continuation and bifurcation software, such as AUTO2000 [19], can efficiently automate the entire process, and also compute branches of limit cycle (oscillatory) solutions with varying control parameter. Judicious use of bifurcation analysis along with a few select simulation runs can be very effective in studying stability and control problems in nonlinear flight dynamics. A recent report [17] provides a concise description of the use of continuation methods for the analysis of aircraft flight dynamics; the reader is also referred to the article by Lowenberg [39].

The use of bifurcation methodology as a tool to analyze problems of flight instability has been demonstrated by Guicheteau [29], and applied to the high angle of attack dynamics of the F-14 by Jahnke and Culick [33]. Ananthkrishnan and Sinha [7] devised an extended bifurcation analysis (EBA) methodology which allowed more than one control parameter to be simultaneously varied, thus enabling the analysis of constrained flight maneuvers, such as level turns or zero-sideslip rolls [69]. Avanzini and de Matteis [13] showed the use of bifurcation analysis to study aircraft dynamics with control augmentation. Sibilski [67] used bifurcation and continuation methods to study the flight dynamics of a helicopter with underslung load. Several examples of the use of bifurcation and continuation methods to nonlinear flight dynamics problems have been illustrated by Goman et al. [25]. A more recent review of bifurcation methods as applied to aircraft dynamics is given by Sinha [68].

Besides the analysis of complete aircraft dynamics, bifurcation methods have been utilized to study particular problems of nonlinear flight dynamics in depth. Continuation algorithms have been especially helpful in efficiently computing and characterizing limit cycle oscillations in terms of their onset, stability, amplitude, and frequency. For example, limit cycles in pitching motion were investigated by Davison et al. [18], while lateral-directional limit cycle dynamics, called wing rock, were studied by Liebst [36]. Coupled lateral-longitudinal limit cycles, called large-amplitude wing rock, which had been observed during the analysis of a model fighter aircraft [56], were explained using bifurcation theory by Ananthkrishnan and Sudhakar [6]. Oscillatory pitch bucking motion and spin limit cycles in a model of the F-18 High Alpha Research Aircraft (HARV) aircraft were captured by a bifurcation analysis carried out by Raghavendra et al. [62].

The bifurcation methodology has also been employed as a tool for the design of flight control systems [26]. Ananthkrishnan and Sudhakar [4] devised nonlinear aileron-rudder interconnect laws using bifurcation studies of inertia-coupled roll maneuvers. Bifurcation methods have been found to be an aid in designing nonlinear dynamic inversion control laws by Littleboy and Smith [37]. Bifurcation tailoring of steady states to avoid unstable flight trims and to promote desirable solutions has been demonstrated by Charles et al. [16]. Raghavendra et al. [62] used the extended bifurcation analysis procedure to identify level flight trim states for use as part of their spin recovery algorithm. Bifurcation methods have also been shown to be valuable in automating the computation of gain schedules for traditional aircraft flight control laws by Sinha et al. [70], and in analyzing aircraft flight control laws based on eigenstructure assignment [24]. In recent years, bifurcation analysis is proving to be a useful tool in the aircraft design, simulation, testing, and evaluation process [42, 75]. Patel and co-workers [38, 54] showed how results from bifurcation studies can be helpful in piloted simulations for investigating aircraft departures. The use of continuation methods for computing flight mechanics parameters was presented by Pashilkar and Pradeep [53]. One of the most promising applications of bifurcation methods is its use as a tool to facilitate flight control law clearance [40]. Looking into the future, it is apparent that the bifurcation and continuation methodology can be an extremely valuable tool when it is integrated into the aircraft design and analysis cycle [10].

In the present paper, we use a few illustrative examples to describe the state-of-the-art in the use of bifurcation and continuation methods to the analysis of aircraft trim and stability. This paper is not meant to be an exhaustive review of all past work in this field, but instead referred to the papers [25, 68] cited above; nor is it intended to provide a tutorial introduction to the practice of bifurcation analysis to aircraft flight dynamics - that is available in previously cited references [17, 39]. Rather, we focus on a variety of problems in nonlinear flight dynamics where bifurcation and continuation methods have been fruitfully applied to yield effective solutions.
The examples are largely culled from the experience of the present authors; however, we hope that readers will pay attention to the works of other researchers, many of which have been cited in the list of references. As far as possible, only papers appearing in archival sources are referenced; reports and papers in conference proceedings are included only when similar work is unavailable elsewhere or when they are very recent.

**Standard Bifurcation Analysis**

Standard bifurcation analysis presents a global picture of the steady (trim) states of a dynamical system and their stability, as a function of a control variable. Many nonlinear dynamical systems, including aircraft dynamics (see Appendix-A), can be modeled as a set of ordinary differential equations of the form

\[ \dot{x} = f(x, u, p), \]  

(1)

where \( x \) is the vector of state variables, \( u \) is a scalar control variable of interest, and \( p \) is a vector of parameters, e.g., other control variables that may be held fixed. Bifurcation and continuation software, such as AUTO2000 [19], carry out two key tasks as part of their solution procedure:

1. Obtaining the steady (trim) states \( x^* \) as a function of the control parameter \( u \). This is achieved by using a continuation algorithm that solves a set of nonlinear algebraic equations, such as:

\[ f(x, u, p) = 0 \]  

(2)

where \( p \) are parameters held fixed. An introduction to continuation methods is given by Morgan [46], a deeper mathematical exposition is available in Allgower and Georg [3], while Seydel [66] provides a more practical description from the viewpoint of algorithms for use in bifurcation analysis.

2. Evaluating the stability of each of these trim states. This is done by numerically computing the Jacobian matrix,

\[ \left. \frac{\partial f}{\partial x} \right|_*, \]  

(3)

and its eigenvalues, where the subscript ‘*’ denotes the trim point at which the Jacobian matrix is evaluated. Changes in stability, which occur as one or more eigenvalues migrate across the imaginary axis on the complex plane, are registered as potential bifurcation points. Conditions are evaluated at these critical points to identify the type of bifurcation and the follow-up action, if at all, to be taken in each case. The books by Strogatz [74], and Hale and Kocak [31], are recommended as introductions to bifurcation theory.

Typical bifurcations of steady states encountered in aircraft flight dynamics fall into one of the following three categories:

1. **Limit Point (LP):** This corresponds to a single real eigenvalue crossing the imaginary axis at the origin; and can be viewed as two solution branches, one stable and the other unstable, meeting and annihilating each other. From another point of view, it is seen as a solution branch turning back on itself, hence it is also called a turning point or, more technically, a saddle-node bifurcation point. This type of bifurcation is primarily responsible for dynamical behavior such as jump and hysteresis.

2. **Branch Point (BP):** This also corresponds to a single real eigenvalue crossing the imaginary axis at the origin, but at a branch point, typically more than one solution branches intersect and exchange stability. Branch points themselves are of at least two kinds: transcritical bifurcations, where two solution branches intersect and exchange stability, and pitchfork bifurcations, where three branches intersect and exchange stability. The latter may be either supercritical or subcritical. At a transcritical or supercritical pitchfork bifurcation, the system departs from the original stable branch into a new set of stable solutions as it traverses the branch point. However, a subcritical pitchfork bifurcation may lead to jump phenomenon.

3. **Hopf Bifurcation (HB):** This corresponds to a pair of complex conjugate eigenvalues migrating across the imaginary axis, and gives rise to a branch of limit cycles. Hopf bifurcations may be either supercritical or subcritical, depending on whether the limit cycles created are stable or unstable. At a supercritical Hopf bifurcation, the system shows finite-amplitude limit cycle oscillations, however, subcritical Hopf bifurcations may show jump phenomenon to either a steady state or another (stable) limit cycle branch.

Most continuation algorithms are capable of tracing branches of stable and unstable limit cycles as well and determining their stability by computing Floquet multipli-
ers of the appropriate locally linearized systems. These limit cycle branches may themselves undergo bifurcations which are characterized by the migration of one or more of the Floquet multipliers across the unit circle in the complex plane. Various possible bifurcations of limit cycles are: (a) Fold bifurcation, (b) Transcritical and Pitchfork bifurcations, (c) Period-doubling or Flip bifurcation, and (d) Secondary Hopf or Neimark-Sacker bifurcation. Details are available in the book by Nayfeh and Balachandran [49].

Typical results from a standard bifurcation analysis (SBA) of problems in aircraft flight dynamics are illustrated next. The results are presented in the form of bifurcation diagrams in which steady state (trim) values of a state variable $x_i$ are plotted against the control variable $u$. Stable steady states are shown in full lines, unstable ones with dashed lines; branch point bifurcations are marked with empty squares, and Hopf bifurcations with filled squares. Limit points or saddle-node bifurcations are not marked as they are obvious from the turning over of the solution branch. Limit cycles are indicated with circles marking the peak amplitude of oscillation - filled circles denote stable limit cycles and empty circles are used for unstable limit cycles.

**High-α Dynamics of F-18 HARV Model**

Raghavendra et al. [62] used an F-18 HARV model with data over a range of angles of attack from -14 to +90 deg, including asymmetric lateral-directional aerodynamics with elevator deflection. The data were discretized at intervals of 1 deg of angle of attack and provided in tabular look-up form to the AUTO2000 software [19]. Results from their bifurcation analysis are shown in Fig. 1, where elevator deflection is used as the variable control, throttle is constant, and aileron and rudder are held at neutral. Thrust vectoring controls were also switched off for this analysis. It must be noted that with varying elevator deflection and throttle held fixed, the trims computed would, in general, correspond to non-level flights. Unstable trims are seen at very low and very high angles of attack, with stable trims over most intermediate values of angle of attack. Several Hopf bifurcations and a few branch point bifurcations are marked in the figure. The Hopf bifurcations at low and medium angles of attack correspond to the onset and demise of phugoid (approximately 0.3-0.4 rad angle of attack), dutch roll (approximately 0.4-0.45 rad angle of attack) and lateral phugoid (approximately 0.55-0.65 rad angle of attack). Of particular interest are the Hopf bifurcations at high angles of attack, labeled H1 and H2. A branch of unstable limit cycles is seen to originate at H1 which turns over at a fold bifurcation (labeled F1) to form stable limit cycles which represent a pitch bucking mode of oscillation with dominant short period oscillations in pitch. The stable limit cycle branch starting at H2 represents a flat, oscillatory, left spin with a large yaw rate and flight path angle approaching -90 deg. All the equilibrium (steady) spin solutions at high angles of attack are seen to be unstable. The predictions, made by the bifurcation analysis were found to match well with observations on a scaled F-18 HARV model tested in a spin tunnel [21].

**Inertia-Coupled Roll Maneuvers**

Instabilities in inertia-coupled roll maneuvers of aircraft, discovered by Phillips [55], were attributed to a jump phenomenon by Gates and Minka [22]. Schy and Hannah [65] were the first to computationally predict the critical control inputs at which jump would occur, using a reduced set of equations of motion (see Appendix-B). They computed approximate steady states for an airplane in rolling motion, called pseudosteady states (PSS), by neglecting the effect of gravity in the equations in Appendix-B. Though their results were presented in a manner very similar to the bifurcation diagrams of today, they did not employ a continuation algorithm for their work. Young et al. [77] established that the jump phenomenon at onset of roll-coupled instability was primarily driven by inertia coupling, and usually occurred at low enough values of angle of attack to be largely uninfluenced by nonlinearities in the aerodynamics.

Results from a standard bifurcation analysis carried out by Ananthkrishnan et al. [11] of the pseudosteady state (PSS) equations of motion for rolling maneuvers of an
airplane are presented in Fig. 2. Aircraft data for these computations are sourced from Ref. [69]. Stable and unstable PSS roll solutions are seen in the figure for varying aileron deflection as the control input, for five different rudder deflection settings. Settings 4 and 5 show limit points (or saddle-node bifurcations) at which jump phenomenon is likely, whereas no limit points are seen for rudder deflections labeled 1 and 2. The curve with rudder setting labeled 3 shows a branch point (transcritical bifurcation) and marks the boundary between dynamics with and without jump.

Some key inferences can be drawn from the bifurcation diagram in Fig. 2. Firstly, many practical systems, including aircraft flight dynamics, are, in reality, multi-parameter systems. However, a standard bifurcation analysis (SBA) can be carried out by varying only one control input at a time, while holding all other parameters constant. For different constant values of the parameter(s) held fixed, the resulting bifurcation diagram can be vastly different, as seen in the composite diagram of Fig. 2 which shows solutions for five different values of the rudder deflection. Thus, interpreting bifurcation diagrams for multi-parameter systems can be a difficult and non-trivial exercise. More interestingly, the results in Fig. 2 suggest that there are preferred values of rudder deflections for which undesirable dynamical phenomena, such as jumps, may be avoided. In fact, the setting labeled 3 marks the threshold between rudder settings for which jump is impossible and those for which jump is likely. This knowledge can be used to devise aileron-rudder interconnect laws that can avoid jump phenomena in roll-coupled maneuvers for an airplane, as demonstrated by Ananthkrishnan and Sudhakar [4].

Extended Bifurcation Analysis

For many dynamical systems, it is normal practice to study or analyze steady (trim) states that share a common property, for instance, straight and level flight trims for airplanes. These requirements can often be stated in terms of constraints on the state variables, for example, zero flight path angle, zero sideslip, and zero roll angle, for airplane level trims. For another example, zero flight path angle, zero sideslip, and constant load factor, can be used to define airplane level turning flight. In other words, very often, it is a particular maneuver that is of interest, and almost inevitably, more than one control input needs to be employed simultaneously to achieve the maneuver. For the zero-sideslip level turn maneuver cited above, the rudder, aileron, and throttle, all need to be varied together when the elevator is used as the control variable to compute an entire family of trim solutions. The standard bifurcation analysis procedure is incapable of handling constrained steady (trim) states defined in this manner. The Extended Bifurcation Analysis (EBA) was developed by Ananthkrishnan and Sinha [7] to analyze constrained maneuvers.

The EBA is implemented in two steps. In the first step, a set of constraint equations are added to the dynamical equations of the system, as below:

$$\dot{x} = f(x, u, p_1, p_2)$$

$$y = g(x)$$

(4)

where $g$ represents the constraint equations. In order to satisfy the constraints, an equal number of parameters out of $p$, that were previously held constant, now need to be freed. Let $p_1$ represent the set of parameters which are freed, and let $p_2$ be the parameters that are still held fixed. The system in Eq. (4) is then solved using a continuation algorithm to obtain the steady (trim) states $x^*$, and the parameter schedules $p_1$ required to satisfy the constraints $g$, as a function of the control variable $u$. In practice, the vector $p_1$ is obtained for discrete, albeit closely spaced, values of the continuation parameter $u$. Linear interpolation is used to obtain $p_1(u)$ in the interval between two consecutive values of $u$ at which $p_1(u)$ is known.

The second step of the EBA solves the original dynamical equations of the system with the parameter schedules obtained from the first step incorporated, as below:

$$\dot{x} = f(x, u, p_1(u), p_2)$$

(5)
It is to be noted that correct stability information is obtained from the Jacobian matrix computed for the system in Eq. (5). Also important is the fact that steady states of Eq. (5) include not only all the trims constrained by \( g \) in Eq. (4), but also other possible trims not satisfying these constraints. These other trims represent departures from constrained flight for the same control parameter combination, and it is crucial to be able to predict them as well. In case where the constrained trims are unstable, the departure state represents the asymptotic state of the airplane if it is disturbed from the initial unstable trim.

At this point, it should be noted that multi-parameter bifurcation analysis is not a novel concept in itself. For example, Carroll and Mehra [15] studied simple aileron-rudder interconnects where the rudder deflection was scheduled as a function of the aileron. However, the interconnect was designed without any explicit constraints on the rolling maneuver. On the other hand, EBA generates parameter schedules on-line for the specified constraints, and uses these parameter schedules to analyze airplane stability in course of the corresponding maneuvers. Application of the EBA method to the F-18 HARV model and to the inertia-coupled roll maneuver problem are demonstrated next.

**Level, Symmetric Flight Trims of F-18 HARV Model**

Raghavendra et al. [62] computed level, symmetric flight trims using the EBA method for the F-18 HARV model discussed previously. Three constraint equations representing zero flight path angle, zero roll angle, and zero sideslip, were included in the first step of the EBA:

\[
g_1 = \gamma = 0, \quad g_2 = \beta = 0, \quad g_3 = \varphi = 0
\]

Three control parameters, namely, throttle, aileron and rudder deflection, were freed in order to satisfy the constraints in Eq. (6) for varying values of elevator deflection, which was the control variable used for the analysis. The schedules of the three freed control parameters required to maintain level, symmetric flight were obtained as shown in Fig. 3. Note the non-zero values of aileron and rudder required at larger values of elevator deflection due to the aerodynamic asymmetry in the model. Also, throttle requirements greater than the limiting value of 1.0 have been plotted in the figure.

The control schedules in Fig. 3 are used in the second step of the EBA to obtain bifurcation diagrams for level, symmetric flight, and these are shown in Fig. 4. Several Hopf bifurcations and a few branch points indicating points of onset of instability are predicted in Fig. 4. It may be noticed that besides the trims with the imposed constraints, other trims representing departed solutions at certain branch points have also been captured. Three distinct bands of level, symmetric flight trims, labeled A,B,C are seen - A are high angle of attack trims that are physically ruled out as they require more throttle than the maximum available limit of 1.0, whereas a part of band B at moderate angles of attack, and all of band C at lower angles of attack are physically attainable. Bifurcation diagrams such as these are likely to correlate better with observations from flight tests, and should be used in piloted simulation studies. They are also useful, as
Raghavendra et al. [62] demonstrated, to obtain a safe set of trim states to which an airplane can be recovered from a departed flight condition.

Constrained Inertia-Coupled Roll Maneuvers

Ananthkrishnan and co-workers [5, 45, 69] analyzed inertia-coupled roll maneuvers of airplanes with constraints such as zero sideslip or velocity-vector rolls. The constrained roll maneuvers computed by Ananthkrishnan et al. [10] are used here to illustrate the procedure. Constraints are placed on the sideslip and the pitch rate, as follows:

\[ g_1 = \beta = 0, \quad g_2 = \dot{\gamma} + 0.18 = 0 \quad (7) \]

The pseudo steady state (PSS) equations for the aircraft dynamics are solved along with the constraints in Eq. (7). Aileron deflection is used as the control variable, and the schedules of rudder and elevator deflections required to satisfy the imposed constraints are first computed. These are obtained as shown in Fig. 5. The bifurcation diagram of PSS roll rate with varying aileron deflection as the control variable, subject to the parameter schedules in Fig. 5, is reproduced from Ref. [10] in Fig. 6. The straight line through the origin in Fig. 6 is the constrained solution on which two branch points (transcritical bifurcations) may be observed. The transverse lines at the branch points represent departed solutions for which the constraints are no longer satisfied. Fig. 6 also shows PSS roll rates that are obtained when the parameter schedules in Fig. 5 are slightly perturbed. A perturbation of one sign gives the curve labeled 2 which is a stable branch with roll rate saturating for large aileron deflections, whereas a perturbation of the opposite sign yields curve 4 which shows a limit point (saddle-node bifurcation) and is hence susceptible to jump instability.

Applications

This section illustrates examples of problems in nonlinear flight dynamics where bifurcation and continuation methods have been fruitfully applied to yield effective solutions.

Effect of Control Saturation on Aircraft Stability

It is known that control saturation can be a key factor in inducing instability in closed-loop control systems [14]. Studies by Littleboy and Smith [37] on an aircraft model, Hypothetical High Incidence Research Model (HHIRM), with a Nonlinear Dynamic Inversion (NDI) control law have shown elevator saturation with increasing angle of attack command to lead to trim states that are oscillatory unstable. Ghosh and Ananthkrishnan [23] implemented an NDI control algorithm (shown in Fig. 7), similar to the one proposed by Snell et al [73], for the F-18 HARV model discussed earlier.

Bifurcation analysis of the closed-loop aircraft dynamics model in Fig. 7 was carried out using AUTO2000 [19]. The commanded angle of attack \( \alpha_c \) was varied as the continuation parameter while the sideslip angle \( \beta \) and wind-axis roll angle \( \mu \) were commanded to be zero. Thrust
vectoring controls $\delta_{pv}$ and $\delta_{yw}$ were not activated. Fig. 8 shows the angle of attack achieved, and the elevator, aileron and rudder deflections imposed, for various commanded angle of attack values. As expected, the graph for the achieved angle of attack levels off once saturation is encountered. The point where the elevator saturates is clearly seen to correspond to a Hopf bifurcation (marked by a filled square) which leads to an oscillatory instability. In fact, the movement of the eigenvalues of the closed-loop dynamical system can be tracked with varying commanded angle of attack parameter as shown in Fig. 9. At saturation, the closed loop eigenvalues are rapidly relocated to the corresponding open loop eigenvalues, and a (complex conjugate) pair of eigenvalues is seen to end up with positive real part, i.e., moved to the right half complex plane. Thus, bifurcation methods can be an useful aid in evaluating control law designs, as was demonstrated earlier by Littleboy and Smith [37], and Goman and Khramtsovsky [26].

Choice of Rigging Angle for Parafoil-Payload System Gliding Flight

A parafoil-payload system in flight can be modeled as a two-body system connected at a common hinge point [59, 72]. The two bodies are modeled as shown in Fig. 10, where C is the common hinge point. The system is controlled in flight by deflecting the aft outboard sections (typically, one-quarter chord from the trailing edge) of the parafoil, called brakes. Symmetric brakes control the glide angle, while asymmetric brake deflection is used to induce turns.

One of the biggest issues facing the parafoil designer is the correct choice of rigging angle, marked $\mu$ in Fig. 10(a). An incorrect choice of rigging angle can either result in post-stall glides with poor glide ratio or poor flare and turning performance. Typically, a correctly rigged parafoil will have a trim angle of attack within 1-2
deg of its stall angle of attack, emphasizing the importance of getting the correct rigging angle.

Presently, rigging a parafoil is an art, highlighting the designer’s skill and experience, and the best rigging angle is obtained largely by trial and error. The use of bifurcation methods can provide parafoil designers with a systematic procedure to pick the best rigging angle and also predict the system performance and stability. Prakash et al. [57, 58] carried out bifurcation analysis for a parafoil-payload system using aerodynamic data for the parafoil in tabular form over a range of angles of attack from -10 to +80 deg. The equations of motion for the parafoil-payload system have been listed in Appendix-C. Their computed trim states (parafoil angle of attack, system glide angle) for various choices of rigging angle $\mu$ are reproduced in Fig. 11. Clearly, there is a very narrow window of rigging angle of about 1 deg centered around $\mu = 9$ deg for which the parafoil-payload system may be expected to show good glide and flare performance as well as satisfactory stability characteristics. Later simulations by Prakash and Ananthkrishnan [60] confirmed that the choice of $\mu = 9$ deg was indeed optimal. Thus, bifurcation methods can serve as a valuable tool for design optimization studies as well.

**Analytical Instability Criteria Based on Bifurcation Theory**

Bifurcation theory not only provides a useful computational tool to analyze and identify instabilities in aircraft flight dynamics, but can also be used to derive approximate analytical criteria for onset of instability. Traditionally, analytical instability criteria have been based on small-perturbation analysis of the aircraft dynamic modes [61]. Literal approximations to aircraft dynamic modes continue to be of interest to flight dynamicists and are often valuable assets in the early phases of the design process [8, 9]. At the same time, bifurcation-based instability criteria have been derived to predict onset of specific kinds of instability, as illustrated in Table-1.

**Instability in Roll-Coupled Maneuvers**

An approximate analytical criterion for onset of instability in inertia-coupled roll maneuvers was derived by Phillips [55]. Phillips’ criterion (and subsequent modifications of it [30]) attempted to compute the critical roll rates at which an airplane would be susceptible to pitch/yaw divergence. In practice, only the first critical roll rate is of interest. However, Schy and Hannah [65] discovered that Phillips’ critical roll rates did not always match well with the computed roll rates at which jump phenomenon was observed from their PSS analysis.

<table>
<thead>
<tr>
<th>Aircraft Model</th>
<th>Computed by SBA</th>
<th>Philips’ criterion [55]</th>
<th>Mahale’s criterion [43]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aircraft A [70]</td>
<td>1.7 rad/s</td>
<td>2.0 rad/s</td>
<td>1.7 rad/s</td>
</tr>
<tr>
<td>Aircraft B [63]</td>
<td>2.6 rad/s</td>
<td>2.9 rad/s</td>
<td>2.6 rad/s</td>
</tr>
</tbody>
</table>

Table-1: Critical roll rates for onset of jump in inertia-coupled roll maneuvers

![Bifurcation diagram of (a) parafoil trim angle of attack, and (b) system trim glide angle, for different choices of rigging angle $\mu$ [58]](image)
Recently, Mahale and Ananthkrishnan [43] have derived an improved analytical approximation for the critical roll rates, based on the knowledge that jump onset occurs at a saddle-node bifurcation, and using techniques first described in Ref. [8]. Their predictions and those from Phillips’ theory are both compared in Table-1 with the first critical roll rate obtained from an SBA computation of the PSS equations for two sets of aircraft data from two different sources. It is clear that analytical criteria based on bifurcation theory can provide excellent approximations to conditions of instability onset, as is evident by the superior predictions from Mahale’s theory [43].

Onset of Wing Rock Limit Cycles

Wing rock was identified by Ross [64] to be a limit cycle oscillation arising due to an unstable Dutch roll mode. In the language of bifurcation theory, wing rock onset is a Hopf bifurcation phenomenon where a pair of complex conjugate eigenvalues crosses the imaginary axis on the complex plane at $\pm i$ [6]. Traditionally, the dynamic directional instability parameter $C_n \beta d y_n$, derived by Moul and Paulson [47], has been used as an approximate criterion for wing rock onset. However, recent work by Lutze et al. [41] has established that $C_n \beta d y_n$ is only a static directional instability parameter, i.e., $C_n \beta d y_n = 0$ corresponds to a single real eigenvalue at the origin on the complex plane. Nevertheless, the $C_n \beta d y_n$ criterion has on many occasions served as a useful indicator of wing rock onset. This has been explained by Goman [27] as due to the fact that for many aircraft the eigenvalues crossing the imaginary axis at onset of wing rock are seen to meet on the real axis in the right half-plane, separate into two real eigenvalues, following which one of the eigenvalues moves back to the left half-plane, crossing the origin in the process. Only when the crossing of the imaginary axis at $\pm i$ (dynamic instability) is closely followed (in terms of angle of attack as a parameter) by the crossing at the origin (static instability), does $C_n \beta d y_n$ work out to be a good approximation to the dynamic wing rock onset phenomenon.

In recent years, Liebst and Nolan [35] have tried to use bifurcation theory to obtain an approximate criterion for wing rock onset. Ananthkrishnan et al. [12] approximated the aircraft lateral-directional dynamics by an undamped system which happened to be Hamiltonian. Then, using the theory of bifurcations in Hamiltonian systems [76], they were able to derive an exact criterion for a Hamiltonian Hopf bifurcation, which served as an approxima-

| Conclusions |

Bifurcation methods are seen to have gained wide acceptance as a powerful tool for aircraft trim and stability analysis. Over the last two decades, there are several instances where bifurcation and continuation tools have been successfully applied to study nonlinear problems in aircraft flight dynamics, such as wing rock, spin, and inertia-coupled roll maneuvers. In recent years, bifurcation analysis has been shown to be useful in the aircraft design, simulation, testing and evaluation process, especially for flight control systems. The use of bifurcation methods to evaluate the effect of control saturation on closed-loop aircraft stability, and to optimize the rigging angle for gliding flight of a parafoil-payload system, have been illustrated here. It is foreseen that the bifurcation and continuation methodology will prove to be a very valuable tool in the aircraft design and analysis cycle.

Fig.12 Plot of $D_{lat}$ with varying trim angle of attack (AoA) predicting onset of wing rock at 21.5 deg AoA [12]
Acknowledgments

Funding from the Aeronautics Research and Development Board, Govt. of India, and the Aeronautical Development Agency, Bangalore, India, for some of the work described in this paper over the years is gratefully acknowledged, as also useful exchanges with other researchers in the field, especially, Dr T.G. Pai, Dr Mikhail Goman, Dr Yoge Patel, Dr Mark Lowenberg, and Dr Fred Culick.

References


18. Davison, P.M., Lowenberg, M.H. and di Bernardo, M., "Experimental Analysis and Modeling of Limit


27. Goman, M.G., Private communication, Austin, TX, Aug. 2003.


Appendix-A

Aircraft Equations of Motion

Equations of motion for rigid aircraft flight dynamics, from Fan et al. [20]:

\[
\begin{align*}
\dot{V} &= \frac{1}{m} \eta T \cos \alpha \cos \beta - QSC_D - mg \sin \gamma \\
\dot{\alpha} &= q - \frac{1}{\cos \beta} \left( \sin \beta (p \cos \alpha + r \sin \alpha) + \frac{1}{mV} (\eta T \sin \alpha + QSC_L - mg \cos \mu \cos \gamma) \right) \\
\dot{\beta} &= p \sin \alpha - r \cos \alpha + \frac{1}{mV} (mg \sin \mu \cos \gamma + QSC_y - \eta T \cos \alpha \sin \beta) \\
\dot{\rho} &= \frac{I_y - I_z}{I_x} qr + \frac{1}{I_x} QStC_i \\
\dot{q} &= \frac{I_z - I_x}{I_y} rp + \frac{1}{I_y} QScC_m \\
\dot{r} &= \frac{I_x - I_y}{I_z} pq + \frac{1}{I_z} QStC_n \\
\dot{\phi} &= p + q \sin \phi \tan \theta + r \cos \phi \tan \theta \\
\dot{\theta} &= q \cos \phi - r \sin \phi
\end{align*}
\]

where

\[
\begin{align*}
\sin \gamma &= \cos \alpha \cos \beta \sin \theta - \sin \beta \sin \phi \cos \theta - \sin \alpha \cos \beta \cos \phi \cos \theta \\
\cos \mu \cos \gamma &= \sin \alpha \sin \theta + \cos \alpha \cos \phi \cos \theta \\
\sin \mu \cos \gamma &= \cos \alpha \sin \beta \sin \theta - \cos \beta \sin \phi \cos \theta - \sin \alpha \sin \beta \sin \phi \cos \theta
\end{align*}
\]

Appendix-B

Reduced-Order Equations of Motion

The reduced set of equations used for analysis of aircraft rolling maneuvers, from Schy and Hannah [65]:

\[
\begin{align*}
\dot{\alpha} &= q - p\beta + z_a \alpha + (g/V) \cos \phi \cos \theta \\
\dot{\beta} &= p\alpha - r + y_\beta \beta + (g/V) \sin \phi \cos \theta \\
\dot{\rho} &= \frac{I_y - I_z}{I_x} qr + l_\rho \beta + l_p p + l_r r + l_\alpha \delta_a + l_\delta_r \delta_r \\
\dot{q} &= \frac{I_z - I_x}{I_y} rp + m_a \alpha + m_\alpha \dot{\alpha} + m_q q + m_\delta e \delta_e \\
\dot{r} &= \frac{I_x - I_y}{I_z} pq + n_p \beta + n_\rho \beta + n_\delta_a \delta_a + n_\delta_r \delta_r \\
\dot{\phi} &= p + q \sin \phi \tan \theta + r \cos \phi \tan \theta \\
\dot{\theta} &= q \cos \phi - r \sin \phi
\end{align*}
\]
Equations of Motion of the Parafoil - Payload System

Equations of motion of the parafoil-payload system from Prakash and Ananthkrishnan [60]:

\[
\begin{bmatrix}
M_b R_{cb} & 0 & M_p T_p & T_b \\
0 & -(M_p + M_F) R_{cp} & -(M_p + M_F) T_p & -T_p \\
I_b & 0 & 0 & -R_{cb} T_b \\
0 & I_p + I_m & 0 & R_{cp} T_p
\end{bmatrix}
\begin{bmatrix}
\dot{\Omega}_b \\
\dot{\Omega}_p \\
\dot{V}_c \\
\dot{F}_c
\end{bmatrix} =
\begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
B_4
\end{bmatrix}
\]

\(B_1 = F^A_b + F^G_b - \Omega_b \times M_b \Omega_b \times R_{cb}\)

\(B_2 = F^A_p + F^G_p - \Omega_p \times (M_p + M_F) \Omega_p \times R_{cp} + M_F \Omega_p \times T_p V_c - \Omega_p \times M_F T_p V_c\)

\(B_3 = -\Omega_b \times I_b \Omega_b\)

\(B_4 = M^A_p - \Omega_p \times (I_p + I_m) \Omega_p\)

The kinematic equations for the trajectory of point C and the Euler angles of the parafoil and the payload are

\[
\begin{bmatrix}
x_c \\
y_c \\
z_c
\end{bmatrix} =
\begin{bmatrix}
u_c \\
v_c \\
w_c
\end{bmatrix}
\]

\(\dot{\phi}_b =
\begin{bmatrix}
1 & \sin \phi_b \tan \theta_b & \cos \phi_b \tan \theta_b \\
0 & \cos \phi_b & -\sin \phi_b \\
0 & \sin \phi_b / \cos \theta_b & \cos \phi_b / \cos \theta_b
\end{bmatrix}
\begin{bmatrix}
p_b \\
q_b \\
r_b
\end{bmatrix}\)

\(\dot{\theta}_b =
\begin{bmatrix}
0 & \cos \phi_b & -\sin \phi_b \\
0 & \cos \phi_b & -\sin \phi_b \\
0 & \sin \phi_b / \cos \theta_p & \cos \phi_b / \cos \theta_p
\end{bmatrix}
\begin{bmatrix}
p_p \\
q_p \\
r_p
\end{bmatrix}\)